THE ALGEBRA OF GREEN AND MACKEY FUNCTORS

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Abstract.

We investigate various Green and Mackey functor analogs of concepts from the theory of rings and modules. In particular, we consider ideals, chain conditions, Krull dimension, decomposition theorems and completion for these functors. We characterize the Jacobson radical and the prime and maximal ideals of an arbitrary Green functor A. We prove various properties of these ideals. We also investigate the Krull dimension of a commutative Green functor.

We analyze the Green and Mackey functors satisfying various chain conditions. For left-modules over Green functors A satisfying a certain noetherian-like condition we study the analog of the tertiary decomposition theorem. For the case when A is commutative we study the analog of the primary decomposition theorem.

We also give induction theorems for various special types of Green and Mackey functors such as, prime and simple Green functors A, simple left-A-modules, cotertiary and coprimary left-A-modules. We end with an induction theory for the completion of a Green functor in a left ideal.

This work generalizes most of the major topics from classical algebra to the category of Green and Mackey functors.

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Introduction.

The theory of Mackey functors originated in the pioneering works of J.A. Green [G] and A.W.M. Dress [Dr2]. Twenty years ago Green introduced the concept of a Mackey functor as an axiomatic treatment for the representation theory of a finite group G. This concept unified several theories: representation theory (character rings, Burnside rings), G-algebras (group algebra, endomorphism algebra of a G-module), K-theory of classifying spaces and finally cohomology. Green defined a G-functor as a collection of abelian groups $M(H)$ (indexed over the subgroups H of the group G) together with restriction maps, transfer maps and conjugation maps, such that certain axioms are satisfied. In particular, an axiom that generalizes Mackey's formula. We refer to a G-functor as a G-Mackey functor or simply, Mackey functor. A Green functor (or algebra G-functor in Green's original terminology) is a Mackey functor satisfying additional axioms. Particularly, one generalizes the Frobenious reciprocity law. Two years later A.W.M. Dress [Dr2] introduced a different axiomatic approach to Mackey functors. In Dress's approach, a Mackey functor is a pair of functors, one covariant and one contravariant, from the category of finite G-sets to the category of abelian groups, such that certain axioms are satisfied. In his work, Dress showed how Mackey functors can be used as a tool for dealing with the general theory of induced representations.

Since then, the theory of Mackey functors developed in many different directions. Two of them are relevant for the present work.

One direction was taken by Thévenaz and Webb. They approached the subject with methods from the representation theory of G-algebras. Using Green's definition, Thévenaz extended the notions of Brauer quotients and primordial subgroups from the theory of G-algebras to the Mackey functor setting. Using the values of the Brauer quotients at the primordial subgroups a Mackey functor M, Thévenaz constructed the twin functor TM of M. This construction has two advantages. On one hand, Thévenaz's results indicate that a lot of information about a Mackey functor M is encrypted in its twin TM. On the other hand, the functor TM can be treated with methods from representation theory of G-algebras. Using the relation between M and TM, Thévenaz has characterized the simple Green functors (see [T3]), and Thévenaz and Webb have characterized the simple Mackey functors (see [TW1]). For more results in this direction see also [T1] and [T2].

A different direction is entirely categorical and started with H. Lindner's paper [Li]. Using Dress's definition, Lindner points out that one can think of Mackey functors as classical additive functors from a small additive category $\mathcal{C}$ to the category of abelian groups. By analyzing the category $\mathcal{C}$, Lewis [Le1] concluded that there exists a "multiplicative" structure in the category of Mackey functors. In other words one can define the "product" of two Mackey functors, which is again a Mackey functor. Following Lewis, we refer to this product as the $\square$ product. In many respects, this product behaves like a tensor product. Using this multiplicative structure, Lewis defines a Green functor as a Mackey functor with a multiplicative structure. Thus one can think of Green functors as analogs of rings. In this category, the Burnside ring Green functor is the analog of the ring $\mathbf{Z}$ of integers. Lewis introduced analogs of almost every basic concept in ring theory. He characterized the Green fields and he found all the prime ideals of the Burnside ring Green functor.

The present work is a continuation of Lewis's unpublished work [Le1].

Chapter 1 of this work is a basic introduction to the category of Mackey functors. We give all three definitions mentioned above and we give some examples. We use [Le1] to introduce the $\square$ product. Using this product, the multiplication for a Green functor A can be described as a morphism

$$\phi : A \square A \longrightarrow A.$$

For a Green functor A we also introduce the notion of a left-A-module.

In chapter 2 we survey the basic results from induction theory. We work with both Green's approach to induction theory and with Dress's approach. Here we give a version of the Dress Induction Theorem.

In chapter 3 we give a description of the $\square$ product of two Mackey functors in terms of Green's definition of a Mackey functor. Namely, for two Mackey functors M and N, and a subgroup H of G, we give a formula for $(M \square N)(H)$. We also give formulae for the transfer maps, the restriction maps and the conjugation maps for the functor $M \square N$. Lewis [Le2] costructed $M \square N$ in terms of the definition of Mackey functors using finite G-sets. Our construction is an interpretation of the one from [Le2].

For a Green functor A and a left-A-module M, we give a formula which allows us to compute the value at H of the product of a left ideal I of A with a submodule N of M. This formula is relevant for analyzing many concepts such as the prime ideals of a Green functor A, the annihilators of a left-A-module M or the completion of A at a left ideal I.

In chapter 4 we introduce the Brauer quotients and the primordial subgroups of a Mackey functor M. We investigate how these concepts behave under various algebraic and Mackey functor related constructions such as restrictions, inductions, direct sums, epimorphic images and $\square$ products. For a subgroup H of G let $W_G H$ be the Weyl group of H in G. For a $W_G H$-module $\overline{M}$ we introduce Lewis's functor $J_{G/H}(\overline{M})$. We refer to it as the J functor of the $W_G H$-module $\overline{M}$. It follows easily, by the results from [Le3], [Le4] and [T3], that if M is a Mackey functor then Thévenaz's twin functor TM can be identified with the direct sum of the J functors of the various Brauer quotients of M. Here the sum is indexed over a set of representatives for the conjugacy classes of primordial subgroups of M. We state some properties of the J functor and of the twin functor.

In chapter 5 we introduce the external product $a \times m$ of an element a of a Green functor A with an element of m a left-A-module M. Here we think of elements a and m as pairs $(a,\ X)$ and $(b,\ Y)$ where X and Y are finite G-sets, and a and m are elements in $A(X)$ and $M(Y)$, respectively. In this case $a \times m$ is an element of $M(X \times Y)$. This definition of the external product is due to Lewis [Le1]. Based on this definition we introduce several algebraic concepts such as, <u>nilpotent elements</u>, <u>units</u> and <u>zero divisors</u>. Here we underline the Mackey functor terminology in order to make the distinction between, for example, the condition on an element a being nilpotent in $A(X)$ (in the classical sense) and a being <u>nilpotent</u> in A (with respect to the external multiplication).

In chapter 6 we begin a very technical study of an especially well-behaved class of Mackey functors namely the characteristic ones. It turns out that a characteristic Mackey functor M has only one conjugacy class of minimal primordial subgroups. A subgroup H in this class is called characteristic for M. If H is a characteristic subgroup for M, we refer to M as H-characteristic.These functors were first investigated by Lewis (see [Le1]). This class includes simple Mackey functors, prime Green functors and coprimary modules. We introduce the concept of a cocharacteristic subfunctor of a Mackey functor. Given a Mackey functor M and a subfunctor N of M, N is cocharacteristic if the quotient functor M/N is characteristic. A characteristic subgroup for M/N is called cocharacteristic for N. In this case we refer to N as being H-cocharacteristic. The notion of cocharacteristic subfunctor is extremely important since many of the interesting subfunctors of a given Mackey functor are cocharacteristic. For example, the prime and maximal ideals of a Green functor A, the annihilators of simple left-A-modules and the primary submodules of a left-A-module are all cocharacteristic. We show that there exists a one-to-one correspondence between cocharacteristic submodules of a left-A-module M and pairs $(H,\ \overline{N})$ where H is a primordial subgroup for M and $\overline{N}$ is a proper W_GH-invariant submodule of the Brauer quotient of M at H. This submodule is denoted by $M_{(H,\ \overline{N})}$. This result is used in an essential way throughout the remainder of our work. We also prove a couple of induction theorems for characteristic Mackey functors.

In chapter 7 we define the Jacobson radical $Jac(A)$ of a Green functor A as the intersection of the annihilators of all simple left-A-modules. For H a primordial subgroup of A let $\overline{Jac_H}$ be the Jacobson radical of the Brauer quotient of A at H. We show that $Jac(A)$ is the intersection of all cocharacteristic ideals of the form $A_{(H,\ \overline{Jac_H})}$ with H varying over the set of primordial subgroups of A. We show that $Jac(A)$ satisfies the Nakayama lemma. We show that $Jac(A)$ consists precisely of all elements a of A such that, for every b element of A, $1 - a \times b$ is invertible in the ring obtained by evaluating A at the appropriate finite G-set. We show that $Jac(A)$ is the largest ideal J of A such that $J(H) \subseteq Jac(A(H))$ for all subgroups H of G. Unfortunately, $Jac(A)$ does not coincide with the intersection of all maximal ideals of A. We conclude with a discussion of commutative Green functors, A, which are canonically isomorphic to their twin, TA. We show that for such Green functors $Jac(A)$ is the intersection of all maximal ideals of A and that $Jac(A)(H) = Jac(A(H))$ for all subgroups H of G.

In chapter 8 we investigate chain conditions. It turns out that the obvious generalizations of the classical chain conditions to the Mackey functor setting are less well behaved than one would expect. For example finitely generated left-A-modules over noetherian Green functors need not be noetherian. However, some results do still hold. For example, we prove Cohen's theorem. We show that artinian Green functors are very well behaved. In particular, they are noetherian, and their Jacobson radical is nilpotent. Moreover, there are only finitely many conjugacy classes of simple left modules over such Green functors. We also show that if A is commutative, artinian, and has trivial Jacobson radical, then $A(H)$ is semisimple artinian for all subgroups H of G. As a corollary, we show that for such A, $Jac(A)(H) = Jac(A(H))$ for all subgroups H of G. We also investigate a stronger analog of the chain conditions which is much better behaved than the straightforward analogue. We say that a Mackey functor M satisfies a chain condition in the strong (or total) sense if the Mackey functors M_X satisfy this condition for all finite G-sets, X. We give characterization theorems for Green functors, A, and left-A-modules, M, satisfying chain conditions in this strong sense.

In chapter 9 we describe all the prime and maximal ideals of a Green functor A. We also present an induction theorem for prime Green functors. When A is commutative, we obtain that the nilradical of A is the intersection of all prime ideals of A (see [T4], corollary 54.6, p. 511). For commutative Green functors A we define the spectrum of A, Spec(A), and we show that it is compact. We also show that Spec(A) is connected if and only if $A(G)$ has no nontrivial idempotents.

In chapter 10 we assume that A is a commutative Green functor. We investigate a certain localization procedure at multiplicative subsets U of $A(G)$. This technique is due to Thévenaz (see [T3], p. 472). We show that this localization has the expected universality property. We investigate the relation between the defect of A and the defect of the localization of A at various prime ideals of $A(G)$. We also show that there exists a one-to-one correspondence between the prime ideals of the localized Green functor $U^{-1}A$ and the prime ideals P of A such that $P(G)$ is disjoint from U.

In chapter 11 we investigate the Krull dimension of a commutative Green functor A. In the Mackey functor setting, we distinguish two types of containments of prime ideals $P \subseteq Q$. The trivial type is when P and Q have the same cocharacteristic subgroup. This is equivalent to investigating the Krull dimension of the Brauer quotient of A at this common cocharacteristic subgroup. Hence it is a problem which belongs in commutative algebra. The harder situation is when P and Q don't have the same cocharacteristic subgroup. Suppose that P and Q are H- and K- cocharacteristic, respectively. From the induction theorem for prime Green functors from chapter 9, it follows easily that in this case A/Q has integer characteristic $q > 1$, and that H and K are in the same q-tower (see [Dr2] or [Le1] for the definition of a q-tower). We show that in this case there exists a sequence of intermediary prime ideals between P and Q such that their cocharacteristic subgroups are in a refined q-tower between H and K. We also investigate the Krull dimension of a Green functor satisfying certain integrality assumptions.

In chapter 12 we define the notion of a characteristic subgroup of an arbitrary Mackey functor M. A subgroup H is characteristic if M has a non-zero submodule which is H-characteristic. We show that if M is non-zero then M has a characteristic submodule. We introduce the characteristic socle of M and we show that it is essential in M. We investigate the Mackey functors which have only one characteristic subgroup up to conjugation. If H is such a subgroup we refer to M as H-cospecial. If N is a submodule of M such that M/N is H-cospecial, we refer to N as H-special in M. We show that every submodule N of a left-A-module M is a finite intersection of special submodules in M. Hence, we have a special decomposition theorem for Mackey functors. We conclude with an induction theorem for cospecial Mackey functors.

In chapter 13 we analyze the tertiary and primary submodules of a left-A-module M. Here A satisfies a noetherian-like assumption. We show that every cotertiary module is cospecial. From the results from chapter 12 we conclude that the tertiary decomposition theorem holds for left-noetherian A-modules. When A is commutative, we show that coprimary modules are characteristic. As a consequence, it follows that the cotertiary noetherian modules, are exactly the ones whose characteristic socle is coprimary. This explains why cotertiary objects are not coprimary in this category. We end with induction theorems for cotertiary and coprimary modules.

Chapter 14 deals with the completion, $\widehat{A}$, of a Green functor A at a left ideal I of A. Our main result is that the defect of $\widehat{A}$ is the defect of A/I.

1. Definitions, Notations and Terminology.

In this first chapter we give three equivalent definitions of Mackey and Green functors. We start with the most elementary point of view. The original idea behind the first definition is due to Green [G]. The following variant of this definition appears in [T2].

Fix a commutative ring R and a finite group G and denote by $S(G)$ the set of all subgroups of G. For $H \in S(G)$ and $g \in G$, let ${}^gH = gHg^{-1}$. If $H \in S(G)$ and $L,\ K \in S(H)$, let $[L\backslash H/K]$ be a set of representatives of double cosets LhK with $h \in H$. Finally, if $K \leq H \leq G$, let H/K be a set of representatives of cosets hK with $h \in H$.

1.1. THE FIRST DEFINITION.

A *G-Mackey functor M (over R)* is a family of R-modules $\big(M(H)\big)_{H\in S(G)}$, together with three families of R-linear morphisms relating these modules. The families r^H_K : $M(H) \to M(K)$ of *restrictions* and $t^H_K : M(K) \to M(H)$ of *transfers* (or *induction maps*) are indexed on the collection of $H,\ K \in S(G)$, such that $K \leq H$. The family $c_g = c_{g,H} : M(H) \to M({}^gH)$ of *conjugation maps* is indexed on $H \in S(G)$ and $g \in G$. For all $g,\ h \in G$ and $H,\ K,\ L \in S(G)$, these maps must satisfy the following conditions:

(i) if $L \leq K \leq H$, $r^K_L r^H_K = r^H_L,\ t^H_K t^K_L = t^H_L$,

(ii) $r^H_H = t^H_H = id_{M(H)}$,

(iii) $c_{gh} = c_g c_h$,

(iv) if $h \in H,\ c_h : M(H) \to M(H)$ is the identity,

(v) if $K \leq H$, $c_g r^H_K = r^{{}^gH}_{{}^gK}\ c_g$ and $c_g t^H_K = t^{{}^gH}_{{}^gK}\ c_g$,

(vi) (Mackey axiom) if $L, K \leq H$,

$$r^H_L t^H_K = \sum_{h\in[L\backslash H/K]} t^L_{L\cap{}^hK}\ r^{{}^hK}_{L\cap{}^hK}\ c_h.$$

Axioms *(ii)-(iv)* imply that the formula in the Mackey axiom does not depend on the choice of representatives of the double cosets. By axioms *(iii)* and *(iv)*, the group $W_GH = N_G(H)/H$ acts on $M(H)$, via R-linear automorphisms. Hence $M(H)$ is an $R[W_GH]$-module.

A *morphism* of Mackey functors $f : M \to N$ is a family of R-module homomorphisms $f(H) : M(H) \to N(H)$, for $H \in S(G)$, which commute with restriction, transfer and conjugation (in the obvious sense). In particular, since f commutes with conjugation, $f(H)$ is an $R[W_G H]$-map. Denote by $\mathit{Mack}_R(G)$ the category of Mackey functors for G over R. The set of homomorphisms from M to N is written $\mathrm{Hom}_{\mathit{Mack}_R(G)}(M,\ N)$. It is easy to see that $\mathit{Mack}_R(G)$ is an abelian category.

We next introduce the concept of *pairing*. Let $M,\ N,\ L$ be G-Mackey functors over R. A *pairing* is a family of R-linear maps

$$M(H) \times N(H) \longrightarrow L(H); \qquad (x,\ y) \mapsto x \cdot y$$

such that, for $H,\ K \in S(G)$ with $K \leq H$,

(vii) $r^H_K(x \cdot y) = r^H_K(x) \cdot r^H_K(y), \quad x \in M(H),\ y \in N(H),$

(viii) $c_g(x \cdot y) = c_g(x) \cdot c_g(y), \quad x \in M(H),\ y \in N(H),\ g \in G,$

(ix) (Frobenius axiom)

$$t^H_K(x \cdot r^H_K(y')) = t^H_K(x) \cdot y', \quad x \in M(K),\ y' \in N(H).$$
$$t^H_K(r^H_K(x') \cdot y) = x' \cdot t^H_K(y), \quad x' \in M(H),\ y \in N(K).$$

We denote this pairing by $M \times N \longrightarrow L$.

A *Green functor* A for G (over R) is a Mackey functor A together with a pairing $A \times A \longrightarrow A$ such that, for each $H \in S(G)$, the R-linear map $A(H) \times A(H) \longrightarrow A(H)$ makes $A(H)$ an associative R-algebra with unity $1_{A(H)}$ such that

(x) if $K \leq H$, then $r^H_K(1_{A(H)}) = 1_{A(K)}$.

In other words, A is a Mackey functor such that, for all $H \in S(G)$, $A(H)$ is an R-algebra, the restriction maps are algebra homomorphisms (thanks to *(vii)* and *(x)*), and the Frobenius axiom holds. Moreover, *(ix)* implies that the image of t^H_K is a two-sided ideal of $A(H)$. Note that t^H_K is not a ring homomorphism. If $A(H)$ is commutative for all $H \in S(G)$, A is called a *commutative Green functor*.

A morphism f of Green functors is a morphism of Mackey functors such that each $f(H)$ is an unitary R-algebra homomorphism.

The following proposition is an easy consequence of Mackey and Frobenius axioms.

(1.1.1) PROPOSITION ([G]) *Let A be a Green functor over R. If $H,\ L,\ K \in S(G)$ with $K \leq H,\ L \leq H$ and if $a \in A(K),\ b \in A(L)$, then:*

$$t_K^H(a)t_L^H(b) = \sum_{h\in[K\backslash H/L]} t^H_{K\cap {}^hL}\left(r^K_{K\cap {}^hL}(a)\cdot r^{{}^hL}_{K\cap {}^hL}c_h(b)\right).$$

Given a Mackey functor M and a Green functor A, M is called a *left-A-module* if there exists a pairing $A \times M \longrightarrow M$ such that $M(H)$ becomes a unitary left $A(H)$-module via the R-linear map $A(H) \times M(H) \longrightarrow M(H)$. One can define similarly the notion of *right-A-module.* For instance, A is both a left and a right module over itself. A left-A-module M is called an *A-module* if A is a commutative Green functor.

A morphism f of *left-A-modules* is a morphism of Mackey functors such that each $f(H)$ is a homomorphism of left $A(H)$-modules.

A morphism $f : M \longrightarrow N$ is *injective* (resp. *surjective*), if every map $f_H : M(H) \longrightarrow N(H)$ is injective (resp. surjective).

There is also an obvious notion of a *subfunctor* (Mackey subfunctor or Green subfunctor in the respective categories) as well as a notion of a *submodule* of a left module over a Green functor. If A is a Green functor and I is a left-submodule of A (where A is viewed as a left-module over itself), then I is called a *left-ideal* of A. In other words, I is a left-module subfunctor of A such that, for $H \in S(G)$, $I(H)$ is a left-ideal of $A(H)$. Moreover, there is an obvious notion of a quotient functor A/I. If I is a left (right) ideal of A, then A/I is a left (right) A-module. If I is both a left and a right ideal of A, then I is called a *two-sided* ideal of A, or simply an *ideal.* When I is an ideal, the quotient functor A/I is a Green functor.

Let A be a Green functor and let M be a left-A-module. If $(N_i)_{i\in\Gamma}$ are submodules of M, then

$$\sum_{i\in\Gamma} N_i \quad \text{and} \quad \bigcap_{i\in\Gamma} N_i$$

are also left submodules of M. Moreover, if $N_i = I_i$ are left (two-sided) ideals of A, then both $\sum_{i\in\Gamma} I_i$ and $\bigcap_{i\in\Gamma} I_i$ are also left (two-sided) ideals of A.

(1.1.2) EXAMPLE. Let M be a left $R[G]$-module. The fixed point functor FP_M is defined by $FP_M(H) = M^H$, where M^H denotes the set of all H fixed points in M. The restriction map $r_K^H : M^H \longrightarrow M^K$ is the inclusion of fixed points. The transfer map $t_K^H : M^K \longrightarrow M^H$ is the relative trace map, defined by $t_K^H(m) = \sum_{h\in H/K} h\cdot m$, for $m \in M^K$. The conjugation maps are defined by $c_g(m) = g\cdot m$ for $g \in G$ and $m \in M$.

There is a forgetful functor $F : M \to M(1)$ from $Mack_R(G)$ to the category $R[G]$-Mod and in turns out (see [TW1]) that $M \longrightarrow FP_M$ is the right adjoint of F. Moreover, this is a full and faithful functor $R[G]$-Mod$\longrightarrow Mack_R(G)$. Hence $R[G]$-Mod embeds in $Mack_R(G)$. However, the category of Mackey functors is much larger. The Mackey functor FP_M satisfies the additional condition

$$t_K^H r_K^H(m) = |H:K| \cdot m \qquad \text{for } K \leq H \text{ and } m \in M^H.$$

A Mackey functor with this property is called *cohomological* (In [Le1] a Mackey functor with this property is called a *Hecke functor*). In [TW2] it is shown that every cohomological Mackey functor is a quotient of FP_M for some $R[G]$-module M. The above module M can be chosen to be a permutation module.

(1.1.3) EXAMPLE. If M is endowed with the structure of a G-algebra over R (i.e. M is an R-algebra and G acts on M by R-linear automorphisms), then the fixed point functor FP_M from the previous example is in fact a cohomological Green functor for G. In particular, when $M = \mathrm{End}_R(W)$, where W is an $R[G]$-module, we get the Green functor $\mathrm{End}_{R*}(W)$.

(1.1.4) EXAMPLE. Let M be an $R[G]$-module and n be a positive integer. For every subgroup H of G, define $H^n(H,\ M)$ to be the n-th cohomology group of H with coefficients in M (or more precisely in $\mathrm{Res}_H^G(M)$). If $K \leq H$ and $g \in G$, let

$$r_K^H : H^n(H,\ M) \longrightarrow H^n(K,\ M),$$
$$t_K^H : H^n(K,\ M) \longrightarrow H^n(H,\ M),$$
$$c_{g,H} : H^n(H,\ M) \longrightarrow H^n({}^gH,\ M),$$

be the restriction map, the transfer (or corestriction) map and the conjugation map respectively. Then $H^n(-,\ M)$ is a Mackey functor. This is proved for instance in [B]. Moreover, this Mackey functor is cohomological (and this explains the terminology). When $n = 0$, we recover the Mackey functor FP_M from example (1.1.2). Group homology and Tate cohomology also yield Mackey functors.

(1.1.5) EXAMPLE. For $H \in S(G)$, let $R_{\mathbf{C}}(H)$ be the ring of ordinary characters of H (with values in $\mathbf{C}$, the field of complex numbers). The restriction, induction and conjugation maps induce maps

$$r_K^H : R_{\mathbf{C}}(H) \longrightarrow R_{\mathbf{C}}(K),$$
$$t_K^H : R_{\mathbf{C}}(K) \longrightarrow R_{\mathbf{C}}(H),$$
$$c_{g,H} : R_{\mathbf{C}}(H) \longrightarrow R_{\mathbf{C}}({}^gH),$$

making $R_{\mathbf{C}}$ into a Green functor over $\mathbf{Z}$. One can also view $R_{\mathbf{C}}(H)$ as the Grothendieck group of finitely generated $\mathbf{C}[H]$-modules. A similar example is obtained if one works with the ring of Brauer characters. Details can be found in many standard references, for instance [CR].

The previous example can be generalized to many types of Grothendieck group constructions. One of these is the *Burnside ring* Green functor.

(1.1.6) EXAMPLE. The *Burnside ring* $B(H)$ of H is the Grothendieck ring of finite H-sets, with addition given by the disjoint union, and multiplication given by the cartesian product. Restriction, induction and conjugation induce maps making B into a Green functor for G over $\mathbf{Z}$, called the *Burnside functor.* The Burnside functor B is universal in the category of Mackey functors in the same way that the ring $\mathbf{Z}$ is universal in the category of rings; that is, for every Mackey functor M, there exists a unique pairing $B \times M \longrightarrow M$ such that M becomes a *B-module.* Moreover, if A is a Green functor, there exists a unique unitary homomorphism of Green functors $u_A : B \longrightarrow A$. For more information see [tD].

For further examples and properties of Mackey functors defined using this elementary approach, see [G], [T4], [Y1], [Y2].

We now give the second definition of Mackey and Green functors.

1.2. THE SECOND DEFINITION.

Let *G-Set* be the category of finite G-sets. A *Mackey functor* M for G is a pair (M_*, M^*), where M_* :*G-Set*$\longrightarrow$ R-Mod is a covariant functor and M^* :*G-Set*$\longrightarrow$ R-Mod is a contravariant functor, which satisfies the following three conditions:

(i) On objects, $M_*(X) = M^*(X)$, for all $X \in$*G-Set.* We write $M(X)$ for the common value of these functors on the G-set X.

(ii) (Additivity axiom) The two embeddings

$$X \xrightarrow{i_X} X \amalg Y \xleftarrow{i_Y} Y$$

in *G-Set* define an isomorphism

$$M^*(X \amalg Y) \xrightarrow{M^*(i_X) \oplus M^*(i_Y)} M^*(X) \oplus M^*(Y)$$

whose inverse is $M_*(i_X) \oplus M_*(i_Y)$.

(iii) (Mackey axiom) For every pull-back diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X_3 & \xrightarrow{\delta} & X_4 \end{array}$$

in *G-Set*, we have $M^*(\delta)M_*(\gamma) = M_*(\beta)M^*(\alpha)$.

The equivalence between definitions 1.1. and 1.2. can be obtained as follows. Assume that M is a Mackey functor in the second sense. For $H \in S(G)$ let $M'(H) = M(G/H)$. For $K \leq H$, and $g \in S(G)$, let us consider the following maps of G-sets:

$$\pi_K^H : G/K \longrightarrow G/H, \qquad xK \longmapsto xH,$$
$$C_g : G/H \longrightarrow G/{}^gH, \qquad xH \longmapsto (xg^{-1})^gH,$$

Define

$$t_K^H = M_*(\pi_K^H), \qquad r_K^H = M^*(\pi_K^H), \qquad c_g = M_*(C_g).$$

It can be checked that the collection of R-modules $(M'(H))_{H \in S(G)}$ with the restrictions, transfers and conjugations defined above satisfies the axioms of definition 1.1. For a detailed proof, see [T2]. In later sections, when we use approaches 1.1 and 1.2 simultaneously, we adopt the convention that $M(H) = M(G/H)$.

A *morphism of Mackey functors* in this second sense is just a natural transformation of Mackey functors, i.e. $f : M \longrightarrow N$ consists of a family of R-linear maps $f(X) : M(X) \longrightarrow N(X)$, indexed by the objects of *G-Set*, such that this family is a natural transformation $M_* \longrightarrow N_*$ and $M^* \longrightarrow N^*$.

If $M,\ N,\ L$ are Mackey functors, then a *pairing* $M \times N \longrightarrow L$ is a family of R-bilinear maps

$$M(X) \times N(X) \longrightarrow L(X), \qquad (x,\ y) \mapsto x \cdot y,$$

indexed by the objects of *G-Set*, such that, for every morphism of G-sets $\phi : X \longrightarrow Y$,

(iii) $(L^*\phi)(x \cdot y) = (M^*\phi)(x) \cdot (N^*\phi)(y), \quad x \in M(Y),\ y \in N(Y)$.

(iv) (Frobenius axioms)

$$x \cdot (N_*\phi)(y') = (L_*\phi)((M^*\phi)(x) \cdot y'), \quad x \in M(Y),\ y' \in N(X),$$
$$(M_*\phi)(x') \cdot y = (L_*\phi)(x' \cdot (N^*\phi)(y)), \quad x' \in M(X),\ y \in N(Y).$$

Using the equivalence between the first and the second definition, the remaining notions can be defined immediately. A *Green functor* A will be a Mackey functor $A : G\text{-}Set \longrightarrow R\text{-}$ Mod together with a pairing $A \times A \longrightarrow A$ such that, with respect to this pairing, $A(X)$ becomes an associative R-algebra with unit and the maps $A^*\phi = \phi^*$ preserve units. Finally, if A is a Green functor, then a *left-A-module* is a Mackey functor M together with a pairing $A \times M \longrightarrow M$ such that, via this pairing, $M(X)$ becomes a left-$A(X)$-module for all $X \in G\text{-}Set$.

One of the advantages of definition 1.2. is that some interesting Mackey functor constructions are easier to introduce.

(1.2.1) EXAMPLE. Let M be a Mackey functor and $X \in G\text{-}Set$. The assignment

$$M_X : Y \longmapsto M(X \times Y),$$

$$M_X^*(f) = M^*(\mathrm{id}_X \times f) \quad , \quad M_{X\,*}(f) = M_*(\mathrm{id}_X \times f)$$

defines a Mackey functor. If A is a Green functor then A_X is both a Green functor and a two-sided-A-module. Moreover, if M is a left-A-module, then so is M_X.

For $Y \in G\text{-}Set$, the projection map $\pi : X \times Y \longrightarrow Y$ defines two morphisms of Mackey functors

$$\theta^X : M \longrightarrow M_X, \qquad \theta^X(Y) = M^*(\pi) = \pi^*,$$

$$\theta_X : M_X \longrightarrow M, \qquad \theta_X(Y) = M_*(\pi) = \pi_*.$$

If M is a left-A-module, then two maps above are morphisms of left-A-modules. When A is a Green functor the map θ^X is a morphism of Green functors.

The Mackey functors M_X and the two maps θ^X and θ_X (for various finite G-sets X) play a fundamental role in induction theory (see Chapter 2). We refer to [Dr2] and [tD] for more details.

We give now a third definition of Mackey functors which is due to Lindner [Li].

1.3. THE THIRD DEFINITION.

Let C^+ be the category whose objects are finite G-sets and whose morphisms from X to Y are equivalence classes of pairs $(\alpha,\ \beta)$, where $\alpha : V \to X$ and $\beta : V \to Y$ are morphisms of G-sets. We represent a morphism by the diagram

$$\begin{array}{ccc} & V & \\ {\scriptstyle\alpha} \swarrow & & \searrow {\scriptstyle\beta} \\ X & & Y \end{array}$$

Two such diagrams $(\alpha,\ \beta)$ and $(\alpha_1,\ \beta_1)$ are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, "\sigma"] & \\ X & & Y \\ & V_1 \arrow[ul, "\alpha_1"] \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

for some isomorphism σ of G-sets.

The composition of the (equivalence classes) of morphisms $(\alpha,\ \beta)$ from X to Y and $(\gamma,\ \delta)$ from Y to Z is obtained by considering the pull-back of β and γ as indicated in the following diagram.

$$\begin{tikzcd} & & U \arrow[dl] \arrow[dr] & & \\ & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & W \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ X & & Y & & Z \end{tikzcd}$$

The disjoint union of G-sets is both a product and a coproduct in $\mathcal{C}^+$. The set of morphisms from X to Y is a free abelian monoid (using disjoint union of the intermediate G-sets V). It is convenient to make $\mathcal{C}^+$ into an additive category by turning free abelian monoids into free abelian groups (by the usual construction). Denote by $\mathcal{C}$ the additive category obtained in this fashion. This category appears under the name *Burnside category* in [Le1].

We now give the third definition. A Mackey functor M for G over R is simply an additive functor $F : \mathcal{C} \to R$-Mod. For a proof of the equivalence between this definition and the previous two see [T2].

Fix two G-sets X and Y. Denote $\mathrm{Hom}_{\mathcal{C}}\ (X,\ Y)$ by $[X,\ Y]$ for simplicity. The direct product of X and Y is a product in the category *G-Set*, but not in $\mathcal{C}$. Therefore it defines an extra structure on $\mathcal{C}$. This induces in turn a "multiplicative" structure in the category $Mack_R(G)$. This structure was first investigated in [Le1]. In order to define this structure, we need some properties of the category $\mathcal{C}$.

There is an obvious functor from $\mathcal{C}$ to its opposite category $\mathcal{C}^{op}$ which is the identity on objects and sends a map $f = (\alpha,\ \beta)$ to the map $Df = (\beta,\ \alpha)$. Notice that, for any X,

Y and Z in $\mathcal{C}$, there is a natural isomorphism

$$[X \times Y,\ Z] \cong [X,\ DY \times Z]$$

(note the use of D to correct the variance). The above isomorphism implies that $DY \times -$ is right adjoint to $- \times Y$ so that $\mathcal{C}$ is a symmetric monoidal closed category. Occasionally, we will have a pair of functors M and N, one covariant and one contravariant, from $\mathcal{C}$ into some other category and a family of maps

$$\eta : M(X) \longrightarrow N(X)$$

which we assert to be a natural transformation. In any such statement, an application of D to either functor to correct the variance is implicit. In particular, the adjunction isomorphism can be written

$$[X \times Y,\ Z] \cong [X,\ Y \times Z].$$

From the above isomorphism, it follows that taking cartesian products provides a natural pairing of $\mathcal{C}$ into itself which should be thought of as a tensor product. There is one obvious family of examples of Mackey functors, namely the representable functors of the form $[-,\ Y]$ for $Y \in \mathcal{C}$, which are denoted by $B_Y \in Mack_R(G)$. The functor $B_{G/G} = B$ is exactly the Burnside functor. The notation B is customary used in the literature for the Burnside ring functor over $\mathbb{Z}$. However, for the purposes of this work, we use B to denote the Burnside functor over R.

Using [Da], the symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal closed structure on the functor category $Mack_R(G)$. That is, for any two Mackey functors M and N, we have a tensor product-like construction $M \square N$. This construction is commutative and associative (up to natural isomorphisms), and has B as its unit. The functor $- \square N$ has a right adjoint $< N, - >$. For any two Mackey functors M and N, $< M,\ N >$ is given on objects by

$$< M,\ N > (X) = \text{Nat}\ (M,\ N_X), \qquad \text{for } X \in \mathcal{C}.$$

That is, the value of $< M,\ N >$ at X is the set of all the morphisms in $Mack_R(G)$ from M to N_X.

In [Le1] it is shown that $M \square N$ is completely characterized by the following result:

(1.3.1) THEOREM ([Le1])

If L, M, and N are Mackey functors, then there is a one to one correspondence between maps

$$\theta : M \square N \longrightarrow L$$

and pairings

$$\theta : (M,\ N) \longrightarrow L$$

(pairings in the sense mentioned in the second definition of Mackey functors).

For a more sophisticated characterization of $M \square N$, see [Le1].

(1.3.2) LEMMA ([Le1])

For a Mackey functor M and any X and Y in C, there are natural isomorphisms

$$B_X \square M \cong M \square B_X \cong M_X \cong < B_X,\ M >,$$

$$B_X \square B_Y \cong B_{X \times Y},$$

$$< B_X,\ B_Y > \cong B_{X \times Y}.$$

Notice that D must be used repeatedly in order to make sense of the naturality of these isomorphisms. We will generally think of M_X as $B_X \square M$ and therefore adopt the convention that it is covariant in X.

A Green functor consists of a Mackey functor A together with maps:

$$u_A : B \longrightarrow A$$

$$\phi : A \square A \longrightarrow A$$

such that the following diagrams commute:

$$\begin{tikzcd}
A \square A \square A \arrow[r, "\phi \square 1_A"] \arrow[d, "1_A \square \phi"] & A \square A \arrow[d, "\phi"] \\
A \square A \arrow[r, "\phi"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \square A \arrow[r, "u_A \square 1_A"] \arrow[dr, "\cong"'] & A \square A \arrow[d] & A \square B \arrow[l, "1_A \square u_A"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

The unlabeled isomorphisms above are those expressing the fact that B is the unit for $\square$. The Green functor A is said to be commutative if the diagram

$$\begin{tikzcd}
A \square A \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & A \square A \arrow[dl, "\phi"] \\
 & A &
\end{tikzcd}$$

commutes, where τ is the commutativity isomorphism for $\square$.

A left-A-module for a Green functor A consists of a Mackey functor M together with an action map

$$\zeta : A \,\square\, M \longrightarrow M$$

such that the following diagrams commute:

$$\begin{tikzcd} A \,\square\, A \,\square\, M \arrow[r, "\phi \,\square\, 1_M"] \arrow[d, "1_A \,\square\, \zeta"] & A \,\square\, M \arrow[d, "\zeta"] \\ A \,\square\, M \arrow[r, "\zeta"] & M \end{tikzcd} \qquad \begin{tikzcd} B \,\square\, M \arrow[r, "u_A \,\square\, 1_M"] \arrow[dr, "\cong"'] & A \,\square\, M \arrow[d, "\zeta"] \\ & M \end{tikzcd}$$

A right-A-module is defined analogously. If A is commutative, then the two notions coincide. A submodule N of M is just a subfunctor of M closed under the action of A. Note that A is both a left and a right module over itself. One can define similarly the notions of a left, right or two-sided ideal. Homomorphisms of Green functors, and of modules over Green functors, are just maps of Mackey functors making the appropiate diagrams commute. One can also define similarly the notion of an A-algebra. We denote the category of Green functors for G over R by $Green_R(G)$. For $A \in Green_R(G)$ let A-*Mod* and A-*Alg* be the categories of left-A-modules and A-algebras, respectively.

(1.3.3) EXAMPLE ([Le1]). Let M be a right-A-module, and let N be a left-A-module. We set

$$\psi_M : (M \,\square\, A) \,\square\, N \longrightarrow M \,\square\, N, \qquad \psi_M = \zeta_M \,\square\, 1_N,$$

$$\psi_N : M \,\square\, (A \,\square\, N) \longrightarrow M \,\square\, N, \qquad \psi_N = 1_M \,\square\, \zeta_N.$$

Let us consider the map

$$\psi_M - \psi_N : M \,\square\, A \,\square\, N \longrightarrow M \,\square\, N.$$

The Mackey functor Coker $(\psi_M - \psi_N)$ is called *the box product of* M *and* N *over* A, and it is denoted by $M \,\square_A\, N$. If A is a commutative Green functor, then $M \,\square_A\, N$ becomes an A-module in a canonical way.

Notice that if $X \in G$-*Set* then, from lemma (1.3.2), it follows that

$$(M \,\square_A\, N)_X \;\cong\; M_X \,\square_A\, N \;\cong\; M \,\square_A\, N_X. \tag{1.1}$$

In chapter 3 we give a description of $M \,\square_A\, N$ using definition 1.1. For more information about $M \,\square_A\, N$, see [Le1].

(1.3.4) EXAMPLE ([Le1]). Let M be a left-A-module. Suppose that I is a left ideal of A, and that N is a left submodule of M. Let

$$i : I \longrightarrow A, \qquad n : N \longrightarrow M,$$

be the canonical inclusions of I in A and N in M, respectively. The *product of I and N*, denoted by $I \cdot N$, is the image of the map

$$I \square N \xrightarrow{i \square n} A \square N \xrightarrow{\zeta} N \xrightarrow{n} M.$$

$I \cdot N$ is a left submodule of N, hence of M. A description of $I \cdot N$ in terms of the definition 1.1. is given in chapter 3. If I and J are ideals of A, so is $I \cdot J$.

Other examples of Mackey and Green functors using this approach can be found in [Le1].

For a fourth approach to the theory of Mackey functors, see [T2] and [TW2].

2. Induction Theory.

In this chapter we survey the basic results from induction theory. We work with both definition 1.1 and 1.2. Therefore we describe both the elementary approach to induction theory (see [TW2]) and Dress's approach (see [Dr2]). Here, we introduce the notation needed to cover all of the aspects of these approaches which are used in later chapters. One of the most important results of this chapter is theorem (2.9), which is a version of the Dress Induction Theorem (see [Y3]). We conclude with a few examples.

Let M be a G-Mackey functor (over R) and $H \in S(G)$. There is an obvious restriction functor denoted

$$\downarrow_H^G : Mack_R(G) \to Mack_R(H)$$

which is defined by keeping the values of M only on the subgroups of H. The *induction* of Mackey functors is the left adjoint of the restriction functor and is written $\uparrow_H^G$. Yoshida (see [Sa]) proved that induction is also right adjoint to restriction. Using our second definition of Mackey functors, there is an easy description of induction (see [TW1]).

For any finite G-set X, one has

$$(M\uparrow_H^G)(X) = M(X\downarrow_H^G)$$

where $X\downarrow_H^G$ denotes the restriction to H of the G-set X. In the same way, it is straightforward to define induction on morphisms. One can see easily that the restriction of Mackey functors satisfies the similar property

$$(M\downarrow_H^G)(X) = M(X\uparrow_H^G).$$

An explicit description of the value of an induced Mackey functor on a subgroup (in the framework of definition 1.1) is given by the formula

$$(M\uparrow_H^G)(K) = \bigoplus_{g\in[H\backslash G/K]} M(H \cap {}^gK). \tag{2.1}$$

In particular, if we denote the trivial subgroup of G by 1, then $(M\uparrow_H^G)(1) = M(1)\uparrow_H^G$ where the latter denotes the ordinary induction of modules. It follows that the embedding of $R[G]$-Mod in $Mack_R(G)$ from example (1.1.2) behaves well: for an $R[H]$-module M, one has

$(FP_M)\uparrow_H^G \cong FP_{M\uparrow_H^G}$. When M is a Green functor for H, $M\uparrow_H^G$ is a Green functor. In this case, for $K \in S(G)$, $(M\uparrow_H^G)(K)$ is an R-algebra with the product defined componentwise. However, induction is only right adjoint to restriction in the category of Green functors.

A Mackey functor M for G is *projective relative to a subgroup* H if M is a direct summand of some induced Mackey functor $N\uparrow_H^G$. As usual, N can be chosen to be $M\downarrow_H^G$. More generally, M *is projective relative to a family of subgroups* $\mathcal{X}$ or $\mathcal{X}$*-projective* if M is a direct summand of some Mackey functor of the form $\bigoplus_i N_i\uparrow_{H_i}^G$ where $H_i \in \mathcal{X}$. Again one can choose $N_i = M\downarrow_{H_i}^G$ for each $H_i \in \mathcal{X}$.

For $H \in S(G)$ and $\mathcal{X} \subseteq S(G)$, we set

$$[H]_G = \{K \in S(G) \mid K = {}^gH \text{ for some } g \in G\},$$

$$[\mathcal{X}]_G = \{[H]_G \mid H \in \mathcal{X}\},$$

$$\mathrm{Cl}_G(\mathcal{X}) = \{H \in S(G) \mid [H]_G = [K]_G \text{ for some } K \in \mathcal{X}\}.$$

If $H,\ K \in S(G)$, we use the notation $[H]_G \leq [K]_G$ to express the fact that H is *subconjugate to* K, i.e. $H \in \mathrm{Cl}_G(S(K))$. If $\mathcal{X} \subseteq S(G)$, let

$$\mathrm{SCl}_G(\mathcal{X}) = \{H \in S(G) \mid [H]_G \leq [K]_G \text{ for some } K \in \mathrm{Cl}_G\mathcal{X})\}.$$

Finally, let $\mathrm{Max}(\mathcal{X})$ (respectively $\mathrm{Min}(\mathcal{X})$) be the set of maximal (respectively minimal) subgroups of $\mathcal{X}$. In the above notation, we omit the subscript G whenever the group G is understood.

Again let $\mathcal{X} \subseteq S(G)$. If M is projective relative to $\mathcal{X}$, then M is projective relative to $\mathrm{SCl}(\mathcal{X})$. Conversely, if M is projective relative to a subconjugacy closed set $\mathcal{X}$, then M is projective relative to a set $\mathcal{X}^o$, where $\mathcal{X}^o$ is an arbitrary set of representatives of subgroups in $[\mathrm{Max}(\mathcal{X})]$. Thus it suffices to assume that $\mathcal{X}$ is closed under subconjugation. A minimal subconjugacy closed set $\mathcal{X}$ such that M is $\mathcal{X}$ projective is called a *defect set* for M. It is not clear that the defect set is unique. However, if M is a Green functor which is projective relative to $\mathcal{X}$ and $\mathcal{Y}$ (subconjugacy closed), then M is projective relative to $\mathcal{X} \cap \mathcal{Y}$. Therefore the defect set of a Green functor is unique and is denoted by $D(M)$.

We now describe Dress's approach to relative projectivity using the second definition (see [Dr2], [tD]). First notice that, if $Y \in$ *G-Set*, then

$$Y\downarrow_H^G\uparrow_H^G \cong Y \times G/H$$

so that, for a Mackey functor M, one gets

$$(M\downarrow_H^G\uparrow_H^G)(Y) \cong M(Y\times G/H) = M_{G/H}(Y).$$

It follows from the above formula that M is projective relative to $\mathcal{X}$ if and only if M is a direct summand of M_X, where $X = \coprod_{H\in\mathcal{X}} G/H$.

At this point we introduce some more definitions. Let $X \in G\text{-}Set$.

(2.1) DEFINITION. M is called *X-injective* (*X-projective*) if the map θ^X (θ_X) of example (1.2.1) is split-injective (split-surjective) as a morphism of Mackey functors.

(2.2) PROPOSITION ([tD]).

The following assertions are equivalent:

(1) M is X-injective.

(2) M is X-projective.

(3) M is a direct summand of M_X as a Mackey functor.

(2.3) PROPOSITION.

Let $X = \coprod_{H\in\mathcal{X}} G/H$ and $M \in Mack_R(G)$. Then M is X-projective if and only if M is projective relative to $\mathcal{X}$. In particular, if $\mathcal{Y}$ is a defect set for M, and M is X-projective, then $\mathcal{Y} \subseteq \mathrm{SCl}(\mathcal{X})$.

Let $X^0 = G/G$ be a point, and let $X^k = \prod_{i=0}^{k-1} X_i$, for $k \geq 1$. Let $\pi_i : X^{k+1} \longrightarrow X^k$ be the projection which omits the i-th factor, for $0 \leq i \leq k$. If M is a Mackey functor, we have two chain complexes

$$0 \longrightarrow M_{X^0} \xrightarrow{d^0} M_X \xrightarrow{d^1} M_{X^2} \xrightarrow{d^2} \dots \tag{2.2}$$

$$0 \longleftarrow M_{X^0} \xleftarrow{d_0} M_X \xleftarrow{d_1} M_{X^2} \xleftarrow{d_2} \dots \tag{2.3}$$

defined by $d^k = \sum_{i=0}^{k}(-1)^i\pi_i^*$, $d_k = \sum_{i=0}^{k}(-1)^i\pi_{i*}$.

(2.4) PROPOSITION ([tD]).

(1) M_X is always X-injective and X-projective.

(2) If M is X-injective, then the complexes (2.2) *and* (2.3) *are exact.*

In general, for arbitrary Mackey functors M and finite G-sets X, it is hard to determine if M is X-injective. Based on *(2)* of proposition (2.4), Lewis introduced the following concept (see [Le1]).

(2.5) DEFINITION ([Le1]). A Mackey functor M satisfies *X-injective induction* if the sequence $0 \longrightarrow M \xrightarrow{d^0} M_X \xrightarrow{d^1} M_{X^2}$ is exact at M and M_X.

If $X = \coprod_{H \in \mathcal{X}} G/H$ and M satisfies X-injective induction, then computing the entire Mackey functor M reduces to computing $M\downarrow^G_H$, for $H \in \mathrm{Max}(\mathcal{X})$. From *(2)* of proposition (2.4), it follows that, if M is X-projective (X-injective), then M satisfies X-injective induction. In particular, $M(G)$ can be computed from the values of M at K, for $K \in \mathrm{SCl}(\mathcal{X})$. However, if M satisfies X-injective induction, then M is not necessarily X-injective. For example, the functor FP_M from example (1.1.2) satisfies $G/1$-injective induction, but it is not in general $G/1$-projective.

The following results are due to Dress.

(2.6) THEOREM ([Dr2], [tD])

The following assertions are equivalent:

(1) A is X-projective.

(2) All left (right)-A-modules are X-projective.

(3) The map $f_* : A(X) \longrightarrow A(G/G)$, induced by the unique map of G-sets $f : X \longrightarrow G/G$ is surjective.

If $X = \coprod_{H \in \mathcal{X}} G/H$, then *(3)* of the above theorem asserts that A is X-projective if and only if the map

$$\sum_{H \in \mathcal{X}} t^G_H : \bigoplus_{X \in \mathcal{X}} A(H) \longrightarrow A(G) \tag{2.4}$$

is surjective.

(2.7) PROPOSITION ([Dr2], [tD]).

Let X and Y be finite G-sets and let A be a Green functor. Then $A(X) \longrightarrow A(G/G)$ and $A(Y) \longrightarrow A(G/G)$ are surjective if and only if $A(X \times Y) \longrightarrow A(G/G)$ is surjective.

We can now rephrase the definition of a defect set as follows.

(2.8) DEFINITION. A *defect set* for a Mackey functor M is a minimal set $\mathcal{X}$ of subgroups of G closed under subconjugation such that if $D = \coprod_{H \in \mathcal{X}} G/H$, then M is D-projective.

From (2.6) and (2.7) it follows that every Green functor A has a unique defect set $D(A)$. Moreover, if M is a left-A-module, then M is $D(A)$-projective.

An induction theorem for a Mackey functor M is a theorem which computes, or gives some restrictions on, the defect set for M.

One of the celebrated induction theorems for Green functors is the Dress Induction Theorem. Although Dress formulated it in terms of the definition 1.2., we find it more convenient to work with its variant in terms of definition 1.1 (see [Y3]).

Let p be a prime number and let $H \in S(G)$. There exists a unique minimal subgroup H_p of H, such that $H_p \triangleleft H$, and H/H_p is a p-group. Moreover, H_p is invariant under $\mathrm{Aut}(H)$, hence $H_p \triangleleft N_G(H)$. Notice also that, since H_p is invariant under $\mathrm{Aut}(H)$, it follows that $(H_p)_p = H_p$. Let $H_G^p \leq N_G(H_p)$ such that H_G^p/H_p is a p-Sylow subgroup of $W_G H_p$. Notice that H_G^p is defined up to conjugation by elements in $N_G(H_p)$. We can assume that $H \subseteq H_G^p$. When $p = 0$, we set $H_p = H_G^p = H$. The subgroup H_G^p has the following properties (see [Dr2]):

(i) $H_p \triangleleft H \leq H_G^p$,

(ii) If there is a $g \in G$ such that ${}^gH \triangleleft K$ and $K/{}^gH$ is a p-group, then $[H] \leq [K] \leq [H^p]$.

If Π be a set of prime numbers, let Π' be the set of primes $q \notin \Pi$. For every natural number n, we denote the Π part of n by n_Π and the Π' part of n by $n_\Pi^\perp = n/n_\Pi$. When $\Pi = \{p\}$, we let $n_p = n_\Pi$ and $n_p^\perp = n_\Pi^\perp$ be the p-part (respectively $\{p\}'$-part) of n.

For $\mathcal{X} \subseteq S(G)$, let

$$\mathcal{H}_\Pi^G \mathcal{X} = \{K \in S(G) \mid [K_p] \in \mathrm{SCl}_G(\mathcal{X}) \text{ for some } p \in \Pi\}. \tag{2.5}$$

(2.9) DRESS INDUCTION THEOREM.

Let A be a Green functor. Then

$$\sum_{K \in \mathcal{H}_\Pi^G \mathcal{X}} t_K^G A(K) + \bigcap_{H \in \mathcal{X}} \mathrm{Ker}\, r_H^G = |G|_\Pi^\perp A(G). \tag{2.6}$$

Assume that A is a Green functor and Π is a set of primes such that $p \cdot 1_{A(G)}$ is invertible in $A(G)$ whenever $p \mid |G|$ and $p \notin \Pi$. The above theorem suggests that, in order to bound the defect set of such a Green functor A, it suffices to find a set $\mathcal{X}$ of subgroups of G such that the map

$$\bigoplus_{H \in \mathcal{X}} r_H^G : A(G) \longrightarrow \bigoplus_{H \in \mathcal{X}} A(H)$$

is injective. If $\mathcal{X}$ is such a set, then, from theorems (2.8) and (2.5), we conclude that $D(A)$ consists of subgroups from $\mathcal{H}_\Pi^G \mathcal{X}$.

(2.10) EXAMPLES. In each of the following examples, M denotes a Mackey functor and $\mathcal{X}$ denotes a family of subgroups such that M is projective relative to $\mathcal{X}$. In fact, in each case (except example 4), $\mathrm{SCl}_G\mathcal{X}$ is $D(M)$.

(1) M is the character ring functor over $\mathbf{Z}$ of example (1.1.5), and $\mathcal{X}$ is the set of elementary subgroups (Brauer's induction theorem).

(2) M is the character ring functor over $\mathbf{Q}$ of example (1.1.5), and $\mathcal{X}$ is the set of cyclic groups (Artin's induction theorem).

(3) M is the Burnside ring functor over any ring R of example (1.1.6), and $\mathcal{X}$ is the set of all subgroups of G (an uninteresting example from the point of view of induction theorems).

(4) $M = H^n(-, V)_p$ is the p-part of the n-th cohomology functor (example (1.1.4)), and $\mathcal{X} = \{Q\}$, where Q is a p-Sylow subgroup of G. When V is the trivial module, the defect set of M is the set of all p-subgroups of G. Otherwise it depends on V.

(5) $M = \widehat{B}$, where B is the Burnside ring functor over any ring R of example (1.1.6), I is the augmentation ideal, and M is the completion of B in I. Then $\mathcal{X}$ is the set of all Sylow subgroups of G. For more details, see [MM] and example (14.7).

For a treatement of induction theory using definition 1.3, see [Le1].

3. The Box Product of Two Mackey Functors.

Let M and N be two Mackey functors for G over R. The sole purpose of this chapter is to describe $M \,\square\, N$ in terms of definition 1.1. A description of $M \,\square\, N$ using definition 1.3 was given in chapter 1. In [Le2], Lewis has described $M \,\square\, N$ in terms of definition 1.2. Our description is an interpretation of the construction from [Le2] in the framework of definition 1.1.

This chapter has two parts. In 3.1 we describe $M \,\square\, N$. Assuming that the object that we construct here is a Mackey functor, we show that it satisfies the universality property (1.3.1). We describe the correspondence between maps $\theta : M \,\square\, N \longrightarrow L$ and pairings $\theta : (M,\ N) \longrightarrow L$ (in the sense mentioned in definition 1.1.). In the end of this part, we outline the construction of $M \,\square_A\, N$ of example (1.3.3). As a corollary, we describe the product between a left ideal I of A and a left submodule N of M (see example (1.3.4)) in terms of definition 1.1. In 3.2 we prove that the object constructed at 3.1 is a Mackey functor. The reader who is not interested in the actual proof can skip 3.2 since the arguments used there are not needed in the subsequent chapters.

3.1. THE CONSTRUCTION OF M $\square$ N.

For $H \in S(G)$, let

$$T(H) = \bigoplus_{K \in S(H)} M(K) \otimes N(K), \tag{3.1}$$

Let $I(H)$ be the R-submodule of $T(H)$ generated by the following elements:

$$\begin{aligned} r_L^K(x) \otimes y' - x \otimes t_L^K(y'), &\quad \text{for } x \in M(K),\ y' \in N(L),\ L \subseteq K \subseteq H, \\ x' \otimes r_L^K(y) - t_L^K(x') \otimes y, &\quad \text{for } x' \in M(L),\ y \in N(K),\ L \subseteq K \subseteq H, \\ \big(c_h(x)\big) \otimes y - x \otimes \big(c_h^{-1}(y)\big), &\quad \text{for } x \in M(K),\ y \in N({}^hK),\ h \in H. \end{aligned} \tag{3.2}$$

(3.1.1) THEOREM.

$$(M \,\square\, N)(H) = T(H)/I(H). \tag{3.3}$$

We set

$$x \,\square_H\, y = x \otimes y \pmod{I(H)}, \qquad \text{for } x \in M(K),\ y \in N(K),\ K \subseteq H. \tag{3.4}$$

It is clear that the elements $x \,\square_H\, y$, for $K \subseteq H$, $x \in M(K)$, $y \in N(K)$, generate $(M \,\square\, N)(H)$ as an R-module. Using relations (3.2), we conclude that the above generators satisfy the following relations

$$\begin{aligned} (x_1 + x_2) \,\square_H\, y &= x_1 \,\square_H\, y + x_2 \,\square_H\, y, \\ x \,\square_H\, (y_1 + y_2) &= x \,\square_H\, y_1 + x \,\square_H\, y_2, \\ (rx) \,\square_H\, y &= x \,\square_H\, (ry), \end{aligned} \tag{3.5}$$

for x, x_1, $x_2 \in M(K)$, y, y_1, $y_2 \in N(K)$, $K \subseteq H$, and $r \in R$, and the relations

$$\begin{aligned} r_L^K(x) \,\square_H\, y' &= x \,\square_H\, t_L^K(y'), && \text{for } x \in M(K),\ y' \in N(L),\ L \subseteq K \subseteq H, \\ x' \,\square_H\, r_L^K(y) &= t_L^K(x') \,\square_H\, y, && \text{for } x' \in M(L),\ y \in N(K),\ L \subseteq K \subseteq H, \\ \big(c_h(x)\big) \,\square_H\, y &= x \,\square_H\, \big(c_h^{-1}(y)\big), && \text{for } x \in M(K),\ y \in N({}^hK),\ h \in H. \end{aligned} \tag{3.6}$$

These are the only relations among the generators $x \,\square_H y$ of $(M \,\square\, N)(H)$.

The conjugation maps for $M \,\square\, N$ are defined as follows. For $g \in G$, let $C_g : T(H) \longrightarrow T({}^gH)$ be the map given on the generators by

$$C_g(x \otimes y) = c_g(x) \otimes c_g(y), \qquad \text{for } x \in M(K),\ y \in N(K),\ K \subseteq H \tag{3.7}$$

It is clear that the C_g's are R-linear isomorphisms and that $C_{gh} = C_g \cdot C_h$ for all g, $h \in G$. Moreover, using axioms *(ii)*, *(iii)* and *(v)* of 1.1., it follows that $C_g(I(H)) = I({}^gH)$. In particular, C_g induces an R-linear isomorphism,

$$(M \,\square\, N)(H) = T(H)/I(H) \xrightarrow{c_g} T({}^gH)/I({}^gH) = (M \,\square\, N)({}^gH),$$

such that

$$x \,\square_H\, y \xmapsto{c_g} c_g(x) \,\square_{{}^gH}\, c_g(y). \tag{3.8}$$

The last relation (3.2) guarantees that the action of c_h on $T(H)/I(H)$ is trivial if $h \in H$. Hence $T(H)/I(H)$ is an $R[W_GH]$-module.

The transfer maps for $M \,\square\, N$ are obtained as follows. Let $J \subseteq H$. Let $i_J^H : T(J) \rightarrow T(H)$ be the canonical inclusion of $T(J)$ in $T(H)$ as a direct summand. Viewing $T(J)$ as an R-submodule of $T(H)$ it is easy to see that $i_J^H(I(J)) \subseteq I(H)$. Hence, i_J^H induces the map

$$t_J^H : T(J)/I(J) \longrightarrow T(H)/I(H),$$

such that

$$x \,\Box_J\, y \overset{t_J^H}{\longmapsto} x \,\Box_H\, y, \quad \text{for } x \in M(L),\ y \in N(L),\ L \subseteq J. \tag{3.9}$$

Finally, the restriction maps for $M \,\Box\, N$ are obtained as follows. First consider the R-linear map:

$$R_J^H : T(H) \longrightarrow T(J)/I(J)$$

given on the generators by

$$x \otimes y \overset{R_J^H}{\longmapsto} \sum_{h \in [J\backslash H/K]} r^{{}^hK}_{J\cap {}^hK}(c_h(x)) \,\Box_J\, r^{{}^hK}_{J\cap {}^hK}(c_h(y)), \quad \text{for } x \in M(K),\ y \in N(K),\ K \subseteq H. \tag{3.10}$$

It is easy to show that R_J^H is a well defined R-linear map. It can be shown that $I(H) \subseteq \text{Ker } R_J^H$, hence R_J^H induces a map r_J^H making the diagram

$$\begin{tikzcd} T(H) \arrow[r, "R_J^H"] \arrow[d, "\pi"'] & T(J)/I(J) \\ T(H)/I(H) \arrow[ur, "r_J^H"'] & \end{tikzcd}$$

commute where π is the canonical projection; that is,

$$r_J^H(x \,\Box_H y) = R_J^H(x \otimes y), \qquad \text{for } L \subseteq H,\ x \in M(L),\ y \in N(L), \tag{3.11}$$

Assume that T/I is a Mackey functor. In order to show that T/I is $M \,\Box\, N$ we use theorem (1.3.1). Hence it is enough to show that there exists a one-to-one correspondence between maps $\theta : M \,\Box\, N \longrightarrow L$ and pairings $\theta : (M, N) \longrightarrow L$. Indeed, if θ is a morphism $M \,\Box\, N \longrightarrow L$, define the pairing:

$$M(H) \times N(H) \longrightarrow L(H) \quad (x,\ y) \longmapsto \theta_H(x \,\Box_H\, y). \tag{3.18}$$

The fact that this is a pairing follows from the fact that θ is a morphism of Mackey functors. Conversely, for any pairing

$$M(H) \times N(H) \longrightarrow L(H) \quad (x,\ y) \longmapsto (x \cdot y)$$

define

$$\theta_H : (M \square N)(H) \longrightarrow L(H)$$

by

$$x \square_H y \longmapsto t_L^H(x \cdot y), \quad \text{for } x \in M(L),\ y \in N(L),\ L \subseteq H. \tag{3.19}$$

The fact that this map is well defined and is indeed a morphism of Mackey functors can be checked easily. Moreover, it is clear that these two assignments are inverse to one another.

Now let A be a Green functor. Let M be a right-A-module and N be a left-A-module. We give an elementary description of $M \square_A N$ of example (1.3.3). Let $H \in S(G)$. Let

$$T_A(H) = \bigoplus_{K \in S(H)} M(K) \otimes_{A(K)} N(K), \tag{3.12}$$

and let $I_A(H)$ be the R-submodule of $T_A(H)$ generated by the following elements:

$$\begin{array}{ll} r_L^K(x) \otimes_{A(L)} y' - x \otimes_{A(K)} t_L^K(y'), & \text{for } x \in M(K),\ y' \in N(L),\ L \subseteq K \subseteq H, \\ x' \otimes_{A(L)} r_L^K(y) - t_L^K(x') \otimes_{A(K)} y, & \text{for } x' \in M(L),\ y \in N(K),\ L \subseteq K \subseteq H, \\ (c_h(x)) \otimes_{A({}^hK)} y - x \otimes_{A(K)} (c_h^{-1}(y)), & \text{for } x \in M(K),\ y \in N({}^hK),\ h \in H. \end{array} \tag{3.13}$$

(3.1.2) THEOREM.

$$(M \square_A N)(H) = T_A(H)/I_A(H). \tag{3.14}$$

The conjugation, transfer and restriction maps for $M \square_A N$ are defined using formulas similar to (3.8), (3.9), (3.10) and (3.11) above.

(3.1.4) COROLLARY.

(1) Let A be a Green functor and let M be a left-A-module. Suppose that I is a left ideal of A and that N is a left submodule of M. Then:

$$(I \cdot N)(H) = I(H) \cdot N(H) + \sum_{K<H} t_K^H(I(K) \cdot N(K)), \quad \text{for all } H \in S(G). \tag{3.15}$$

In particular, $I \cdot N$ is the smallest submodule N' of M such that $N'(H) \supseteq I(H) \cdot N(H)$ for all $H \in S(G)$.

(2) Let $(I_i)_{i=1}^n$ be a set of left (two-sided) functorial ideals of A. Then

$$\Big(\prod_{i=1}^n I_i\Big)(H) = \prod_{i=1}^n I_i(H) + \sum_{K<H} t_K^H\Big(\prod_{i=1}^n I_i(K)\Big), \quad \text{for all } H \in S(G). \tag{3.16}$$

In particular, $\prod_{i=1}^{n} I_i$ is the smallest left (two-sided) ideal I' of A such that $I'(H) \supseteq \prod_{i=1}^{n} I_i(H)$ for all $H \in S(G)$.

(3) Let I be a functorial ideal such that $I(H)$ is nilpotent for every $H \in S(G)$. Assume, for example, that, for some $n > 0$, $\big(I(H)\big)^n = 0$. Then $I^n = 0$.

PROOF. *(1)* We use the notation from example (1.3.4). The R-module $(I \,\square\, N)(H)$ of $(A \,\square\, M)(H)$ is generated by the elements $x \,\square_H\, y$ for $K \in S(H)$, $x \in I(K)$, and $y \in N(K)$. It follows that the image of the map $\big(\zeta \circ (i \,\square\, n)\big)(H) = (I \cdot N)(H)$ is generated by the elements

$$\big(\zeta \circ (i \,\square\, n)\big)(H)(x \,\square_H\, y) = \zeta(H)(x \,\square_H\, y) = t_K^H(x \cdot y).$$

Formula (3.14) follows immediately.

(2)-(3) Immediate consequences of *(1)*. $\triangle$

3.2. THE PROOF.

We prove only theorem (3.1.1). We show that T/I, with the conjugation, transfer and restriction maps defined at 3.1. is a Mackey functor. We check that T/I satisfies axioms *(i)-(v)* from 1.1. Notice that *(ii)* and *(iii)* are immediate, and *(iv)* follows due to the last line in formula (3.2). Notice also that the second part of *(i)* is immediate. The only hard axioms to prove are the first part of *(i)* and *(v)*. At this point we find it more convenient to think in terms of G-sets. We use the following lemma:

(3.2.1) LEMMA ([ML]).

Let C a category with pullbacks. Consider the following diagram:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet. \end{tikzcd}$$

(a) If both squares are pullbacks, then the outside rectangle is a pullback.

(b) If the outside rectangle and the right-hand square are pullbacks, so is the left square.

For the proof of the first part of *(i)*, assume $L \subseteq K \subseteq H$, and let $x \,\square_H\, y \in T(H)/I(H)$

for some $x \in M(J)$, $y \in N(J)$, and $J \subseteq H$. Consider the following diagram

$$\begin{tikzcd}
X = \coprod_i G/L_i \arrow[r, "\sqcup l_i"] \arrow[d, "\sqcup m_i"'] & G/L \arrow[d, "\pi_1"] \\
Y = \coprod_j G/J_j \arrow[r, "\sqcup j_j"] \arrow[d, "\sqcup k_i"'] & G/K \arrow[d, "\pi_2"] \\
G/J \arrow[r, "\pi"] & G/H
\end{tikzcd} \tag{3.17}$$

where $\pi,\ \pi_1,\ \pi_2$ are the canonical morphisms of G-sets. Notice that

$$r_L^H(x \,\square_H\, y) = \sum_{h\in[L\backslash H/J]} r^{hJ}_{L\cap {}^hJ}(c_h(x)) \,\square_L\, r^{hJ}_{L\cap {}^hJ}(c_h(y)), \tag{3.18}$$

and that $X = \coprod_{h\in[L\backslash H/J]} G/L \cap {}^hJ$, is the G-set from the top left corner of the diagram (3.17).

Similarly, we have:

$$r_L^K r_K^H(x \,\square_H\, y) = \sum_{h\in[K\backslash H/J]} r_L^K\left(r^{hJ}_{K\cap {}^hJ}(c_h(x)) \,\square_K\, r^{hJ}_{K\cap {}^hJ}(c_h(y))\right)$$

$$= \sum_{h\in[K\backslash H/J]} \sum_{k\in[L\backslash K/K\cap {}^hJ]} r^{khJ}_{L\cap {}^k(K\cap {}^hJ)}(c_{kh}(x)) \,\square_L\, r^{khJ}_{L\cap {}^k(K\cap {}^hJ)}(c_h(y)). \tag{3.19}$$

Notice that $Y = \coprod_{h\in[K\backslash H/J]} G/(K \cap {}^hJ)$ is the G-set from the diagram (3.17), and

$$\coprod_{h\in[K\backslash H/J]} \coprod_{k\in[L\backslash K/(K\cap {}^hJ)]} G/(L\cap {}^k(K\cap {}^hJ))$$

is exactly the pullback of Y and G/L along G/K which, according to lemma (3.2.1) coincides with X. This shows that (3.18) and (3.19) are equal; hence the first part of *(i)* follows.

Similar reasoning can be employed to obtain *(v)*. Finally, the assertion that $I(H) \subseteq \operatorname{Ker} R_K^H$ can be proved using a similar argument.

4. Primordial Subgroups.

In this chapter we introduce the Brauer homomorphisms for a Mackey functor M. This notion is a generalization to the Mackey functor setting of the Brauer homomorphism for a G-algebra (see [T4], p. 91). Associated with the Brauer homomorphism there is a natural notion of a Brauer quotient and of a primordial subgroup. The primordial subgroups of a Mackey functor M and the corresponding Brauer quotients are fundamental for the analysis of the lattice of subfunctors of M. In later chapters we show that these two sets of data encrypt all the information that is needed in order to characterize some of the interesting subfunctors of a Mackey functor (such as prime and maximal ideals of a Green functor A, simple left-A-modules and the Jacobson radical of A).

This chapter has two parts. Throughout most of 4.1 we will adopt the approach from [T1]. Hence, we work with definition 1.1. In this section, we introduce the notions of primordial subgroup and Brauer homomorphism, and we prove most of their properties which are needed in later sections. In particular, we investigate how these concepts behave under various Mackey functor related constructions such as restrictions, inductions, direct sums, epimorphic images and box products. We show that there is a strong relationship between $D(A)$ and the primordial subgroups of a Green functor A. We conclude this section with a characterization of the primordial subgroups in terms of definition 1.2. In part 4.2 we construct the twin functor TM of a Mackey functor M using the values of the Brauer quotients for the various primordial subgroups of M. We give two alternative descriptions of TM. We also construct a canonical morphism β_M from M to TM using the various Brauer homomorphisms. We investigate various properties of β_M and TM which will be relevant in later chapters. The results from 4.2 can be found in [T1], [T3], [Le3] and [Le4].

4.1. PRIMORDIAL SUBGROUPS AND BRAUER QUOTIENTS.

(4.1.1) DEFINITION. Let M be a G-Mackey functor over R. For all $H \in S(G)$, let

$$Tr_{_M}(H) = \sum_{K<H} t^H_K(M(K)), \qquad \text{and} \qquad \overline{M(H)} = \frac{M(H)}{Tr_{_M}(H)}. \tag{4.1}$$

If $H = 1$ we adopt the convention that $Tr_{_M}(1) = 0$ and $\overline{M(1)} = M(1)$. The R-module $\overline{M(H)}$ is called the *Brauer quotient* (of M at H). The canonical epimorphism br^M_H : $M(H) \longrightarrow \overline{M(H)}$ is called *the Brauer homomorphism* (of M at H). A subgroup $H \in S(G)$ such that $\overline{M(H)} \neq 0$ is called a *primordial* subgroup of M. The set of primordial subgroups of M is denoted by $\mathcal{P}(M)$.

The R-submodule $Tr_{_M}(H)$ is invariant under conjugation by $N_G(H)$ because ${}^g(t^H_K(M(K))) = t^H_{{}^gK}(M({}^gK))$ if $g \in N_G(H)$ (by axiom *(v)* of definition 1.1.). Thus br^M_H is a homomorphism of $R[W_GH]$ modules. When the context is clear, we often write br_H instead of br^M_H.

If A is a Green functor for G over R, then $t^H_K(A(K))$ is an ideal of $A(H)$ (by the Frobenius axiom), and therefore $Tr_{_A}(H)$ is an ideal. It follows that $\overline{A(H)}$ is an R-algebra and that the Brauer homomorphism $br^A_H : A(H) \longrightarrow \overline{A(H)}$ is a homomorphism of $R[W_GH]$ algebras.

We begin our analysis with the following lemma.

(4.1.2.) LEMMA.

Let M be a Mackey functor. Then $\mathcal{P}(M)$ is closed under conjugation. Moreover, M is non-zero if and only if $\mathcal{P}(M) \neq \emptyset$. In this case, if $H \in \mathrm{Min}(\mathcal{P}(M))$, then $\overline{M(H)} = M(H)$.

PROOF. It is clear that $\mathcal{P}(M)$ is closed under conjugation. If $M \neq 0$ then let $H \in S(G)$ such that $M(H) \neq 0$. If we choose H minimal with this property it follows easily that $H \in \mathrm{Min}(\mathcal{P}(M))$. It is obvious that $\overline{M(H)} = M(H)$ for such subgroups H. Conversely, it is obvious that $M \neq 0$ if $\mathcal{P}(M) \neq \emptyset$. $\triangle$

(4.1.3) PROPOSITION.

Let M be a Mackey functor and A be a Green functor. Then:

(1) $\overline{(M\downarrow^G_H)(K)} = \overline{M(K)}$ for all $K \in S(H)$. In particular, $\mathcal{P}(M\downarrow^G_H) = \mathcal{P}(M) \cap S(H)$.

(2) If M is a left-A-module, then $\mathcal{P}(M) \subseteq \mathcal{P}(A)$. Moreover, $\overline{M(H)}$ is a left-$\overline{A(H)}$-module for all $H \in \mathcal{P}(M)$.

(3) If $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ *is an exact sequence of Mackey functors, then*

$$\mathcal{P}(L) \subseteq \mathcal{P}(M) \subseteq \mathcal{P}(L) \cup \mathcal{P}(N). \tag{4.2}$$

Moreover, the epimorphism $M \longrightarrow L \longrightarrow 0$ *induces an epimorphism* $\overline{M(H)} \longrightarrow \overline{N(H)} \longrightarrow 0$ *for all* $H \in S(G)$.

(4) If $(M_i)_{i\in I}$ *are Mackey functors, then* $\mathcal{P}(\bigoplus_{i\in I} M_i) = \bigcup_{i\in I} \mathcal{P}(M_i)$. *Moreover,*

$$\overline{(\bigoplus_{i\in I} M_i)(H)} = \bigoplus_{i\in I} \overline{M_i(H)} \quad \textit{for all } H \in S(G). \tag{4.3}$$

PROOF. *(1)* Obvious (see [T1]).

(2) Assume that $\mathcal{P}(A) \neq S(G)$, otherwise there is nothing to prove. If $H \notin \mathcal{P}(A)$, then:

$$1_{A(H)} = \sum_{K\subset H} t_K^H(a_K), \qquad \text{for some } a_K \in A(K).$$

Hence, for $m \in M(H)$ one has

$$m = 1_{A(H)} \cdot m = \sum_{K\subset H} t_K^H(a_K \cdot r_K^H(m)) \in Tr_M(H).$$

Thus $\overline{M(H)} = 0$.

Let now $H \in \mathcal{P}(M)$. In order to show that $\overline{M(H)}$ is a left-$\overline{A(H)}$-module, it suffices to prove that $Tr_A(H) \subseteq \mathrm{Ann}_{A(H)}\overline{M(H)}$. Let $a \in Tr_A(H)$ and $m \in M(H)$. Then

$$a = \sum_{K\subset H} t_K^H(a_K), \qquad \text{for some } a_K \in A(K).$$

Hence

$$a \cdot m = \sum_{K\subset H} t_K^H(a_K) \cdot m = \sum_{K\subset H} t_K^H(a_K \cdot r_K^H(m)) \in Tr_M(H),$$

or $br_H^M(a \cdot m) = 0$.

(3) Let $0 \longrightarrow N \xrightarrow{\psi} M \xrightarrow{\phi} L \longrightarrow 0$ be a short exact sequence. We first show that $\mathcal{P}(L) \subseteq \mathcal{P}(M)$. Assume $H \notin \mathcal{P}(M)$, and $l \in L(H)$. Then

$$l = \phi_H(m) = \phi_H\Big(\sum_{K\subset H} t_K^H(m_K)\Big) = \sum_{K\subset H} t_K^H(\phi_K(m_K)) \in Tr_L(H).$$

Hence $\overline{L(H)} = 0$.

We now show that $\mathcal{P}(M) \subseteq \mathcal{P}(N) \cup \mathcal{P}(L)$. Assume $H \notin \mathcal{P}(N) \cup \mathcal{P}(L)$. We show that $H \notin \mathcal{P}(M)$. Let $m \in M(H)$. Then, since $H \notin \mathcal{P}(L)$, we have

$$\phi_H(m) = \sum_{K \subset H} t_K^H(l_K) = \sum_{K \subset H} t_K^H(\phi_K(m_K)) = \phi_H\Big(\sum_{K \subset H} t_K^H(m_K)\Big).$$

Hence

$$m - \sum_{K \subset H} t_K^H(m_K) \in \text{Ker } \phi_H = \text{Im } \psi_H.$$

Therefore

$$m - \sum_{K \subset H} t_K^H(m_K) = \psi_H(n)$$

for some $n \in N(H)$. However, since $H \notin \mathcal{P}(N)$, we can write

$$n = \sum_{K \subset H} t_K^H(n_K)$$

and

$$\psi_H(n) = \psi_H\Big(\sum_{K \in H} t_K^H(n_K)\Big) = \sum_{K \subset H} t_K^H(\psi_K(n_K)).$$

It follows that

$$m - \sum_{K \subset H} t_K^H(m_K) = \sum_{K \subset H} t_K^H(\psi_K(n_K)).$$

Therefore $m \in Tr_M(H)$. In conclusion $\overline{M(H)} = 0$; hence $H \notin \mathcal{P}(M)$.

Finally, due to the naturality of $\phi(H)$, the surjection

$$M(H) \xrightarrow{\phi(H)} L(H) \longrightarrow \overline{L(H)} \longrightarrow 0$$

factors through $Tr_M(H)$. Hence the sequence $\overline{M(H)} \longrightarrow \overline{L(H)} \longrightarrow 0$ is exact at $\overline{L(H)}$.

(4) Let $M = \bigoplus_{i \in I} M_i$. If I is finite, *(4)* is a immediate consequence of *(3)*. Otherwise, since M_i are epimorphic images of M, it follows immediately from *(3)* that $\mathcal{P}(M_i) \subseteq \mathcal{P}(M)$. Hence $\cup_{i \in I} \mathcal{P}(M_i) \subseteq \mathcal{P}(M)$.

Conversely, assume that $H \notin \cup_{i \in I} \mathcal{P}(M_i)$, and let $x \in M(H)$ be arbitrary. Since this element x is actually in $\bigoplus_{i \in I_1} M_i$, for some finite set $I_1 \subset I$, it follows immediately that

$$x \in \sum_{K \subset H} t_K^H\left(\bigoplus_{i \in I_1} M_i(K)\right) \in Tr_M(H).$$

We conclude that $\overline{M(H)} = 0$. Hence $H \notin \mathcal{P}(M)$.

Formula (4.3) follows immediately from the previous arguments. △

If $N \subseteq M$, there is no general containment relation between $\mathcal{P}(N)$ and $\mathcal{P}(M)$ as is shown by the following example.

(4.1.4) EXAMPLE. Let p be a prime number and let $G = \mathbf{Z}_p$. Let $M(G) = M(1) = \mathbf{Z}_p$, and define the restriction map r_1^G to be the zero map and the transfer map $t_1^G : \mathbf{Z}_p \longrightarrow \mathbf{Z}_p$ to be the identity map of $\mathbf{Z}_p$. One can easily check that M is a Mackey functor. Notice that $\mathcal{P}(M) = \{1\}$. Now M has a subfunctor N such that $N(G) = \mathbf{Z}_p$ and $N(1) = 0$. It is clear that $\mathcal{P}(N) = \{G\}$. Hence there is no containment relation between $\mathcal{P}(M)$ and $\mathcal{P}(N)$.

(4.1.5) COROLLARY.

Let A be a Green functor and $f : M \longrightarrow L$ be a morphism of left-A-modules. Then

(1) f is injective if and only if $f(H)$ is injective for all $H \in \mathcal{P}(A)$.

(2) f is surjective if and only if $f(H)$ is surjective for all $H \in \mathcal{P}(N)$.

PROOF. *(1)* If $f(H)$ is injective whenever $H \in \mathcal{P}(A)$, it follows that $\mathcal{P}(A) \cap \mathcal{P}(\mathrm{Ker}\ f) = \emptyset$. However, according to (4.1.3) *(2)*, $\mathcal{P}(\mathrm{Ker}\ f) \subseteq \mathcal{P}(A)$. We conclude that $\mathcal{P}(\mathrm{Ker}\ f) = \emptyset$; hence $\mathrm{Ker}\ f = 0$.

(2) If $f(H)$ is surjective whenever $H \in \mathcal{P}(L)$, it follows that $\mathcal{P}(L) \cap \mathcal{P}(\mathrm{Coker}\ f) = \emptyset$. However, according to (4.1.3) *(3)*, $\mathcal{P}(\mathrm{Coker}\ f) \subseteq \mathcal{P}(L)$. We conclude that $\mathcal{P}(\mathrm{Coker}\ f) = \emptyset$; hence $\mathrm{Coker}\ f = 0$. △

(4.1.6) PROPOSITION.

Let A be a Green functor, and let M be a left-A-module. Then:

(1)

$$Tr_M(H) = \sum_{\substack{K \in \mathcal{P}(M) \\ K < H}} t_K^H(M(K)) \qquad \text{for all } H \in S(G). \tag{4.4}$$

In particular

$$M(H) = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H(M(K)), \qquad \text{for all } H \in S(G).$$

(2) The map

$$\bigoplus_K r_K^H : M(H) \longrightarrow \bigoplus_{K \in \mathcal{P}(A) \cap S(H)} M(K)$$

is one-to-one.

(3) Let N be a submodule of M. Then

$$N(H) = \bigcap_{K \in \mathcal{P}(A) \cap S(H)} (r_K^H)^{-1}(N(K)). \tag{4.5}$$

PROOF. *(1)* See [T1].

(2) If $H \in \mathcal{P}(A)$, there is nothing to prove. Assume $H \notin \mathcal{P}(A)$. Let $m \in M(H)$ be such that $r_K^H(m) = 0$ for all $K \in \mathcal{P}(A) \cap S(H)$. We show that $m = 0$. Since $H \notin \mathcal{P}(A)$, we use *(1)* to conclude that

$$1_{A(H)} = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H(a_K), \qquad \text{for some } a_K \in A(K).$$

Then

$$m = 1_{A(H)} \cdot m = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H(a_K) \cdot m = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H\big(a_K \cdot r_K^H(m)\big) = 0.$$

(3) Follows from *(2)* applied to the left-A-module M/N. $\triangle$

(4.1.7) PROPOSITION.

Let A be a Green functor. Then $\mathcal{D}(A) = \mathrm{SCl}_G(\mathcal{P}(A))$.

PROOF. See [T1]. $\triangle$

(4.1.8) PROPOSITION ([T1]).

Let M be a cohomological G-Mackey functor over R. Then $\mathcal{P}(M)$ consists of p-groups for primes p which are not invertible in R. In particular, if Π is the set of primes p which are not invertible in R then every cohomological Green functor A for G over R is $\coprod_{p \in \Pi} G/S_p$-projective, where S_p is a p-Sylow subgroup of G.

PROOF. See [T1]. $\triangle$

We now investigate the primordial subgroups and the Brauer quotients of induced Mackey functors. Let $H,\ K \in S(G)$. Let

$$D(H, K) = \{g \in [H \backslash G/K] \mid {}^gK \subseteq H\}. \tag{4.7}$$

(4.1.9) PROPOSITION.

(1) If $M \in Mack_R(H)$, then

$$\mathcal{P}(M\uparrow_H^G) = \mathrm{Cl}_G(\mathcal{P}(M)).$$

(2) If $K \in \mathcal{P}(M\uparrow_H^G)$, then

$$\overline{(M\uparrow_H^G)(K)} = \bigoplus_{\{g\in D(H,K)\ \mid\ {}^gK\in\mathcal{P}(M)\}} \overline{M({}^gK)}.$$

PROOF. *(1)* Let B be the Burnside Green functor for H over R. We first show that

$$\mathcal{D}(B\uparrow_H^G) = \mathrm{SCl}_G(H).$$

Indeed, from formula (2.1), it follows easily that $B(H)$ is a direct summand in $(B\uparrow_H^G)(H)$, and that $(B\uparrow_H^G)(G) = B(H)$. This shows that the map

$$(B\uparrow_H^G)(H) \longrightarrow (B\uparrow_H^G)(G)$$

is surjective. From proposition (2.3) and theorem (2.6), it follows that $\mathcal{D}(B\uparrow_H^G) \subseteq \mathrm{SCl}_G(H)$. Since $M\uparrow_H^G$ is an $B\uparrow_H^G$ module, it follows that

$$\mathcal{P}(M\uparrow_H^G) \subseteq \mathcal{P}(B\uparrow_H^G) \subseteq \mathcal{D}(B\uparrow_H^G) \subseteq \mathrm{SCl}_G(H).$$

Since $\mathcal{P}(M\uparrow_H^G)$ is closed under conjugation, it is enough to show that if $K \in S(H)$, then K is primordial for M if and only if K is primordial for $M\uparrow_H^G$. We first show that

$$\mathcal{P}(M) \subseteq \mathcal{P}(M\uparrow_H^G).$$

Let $K \in \mathcal{P}(M)$. From formula (2.1), we know that

$$(M\uparrow_H^G)(K) = \bigoplus_{g\in[H\backslash G/K]} M(H \cap {}^gK).$$

Since $K \leq H$, it follows that $M(K)$ is the direct summand in $(M\uparrow_H^G)(K)$ corresponding to the double coset HeK, where e is the identity element of the group G. Moreover, if $L < K$ and $g \in G$, then $H \cap {}^gL$ is either incomparable to K or strictly contained in K. Now if $K \notin \mathcal{P}(M\uparrow_H^G)$, then the map

$$\bigoplus_{L<K} (M\uparrow_H^G)(L) \longrightarrow (M\uparrow_H^G)(K)$$

must be surjective. However, this would imply that the map

$$\bigoplus_{g\in G}\ \bigoplus_{\{L<K\ \mid\ H\cap {}^gL<K\}} M(H\cap {}^gL) \longrightarrow M(K)$$

is surjective, contradicting the fact that $K\in\mathcal{P}(M)$.

We now show that

$$\mathcal{P}(M\uparrow_H^G)\cap S(H)\subseteq \mathrm{SCl}_G(\mathcal{P}(M))\cap S(H).$$

Let $K\in\mathcal{P}(M\uparrow_H^G)$, and suppose that $K\leq H$. From formula (2.1), it follows that

$$(M\uparrow_H^G)(K)=\bigoplus_{g\in[H\backslash G/K]} M(H\cap {}^gK).$$

We show that ${}^gK\in\mathcal{P}(M)$ for some $g\in D(H,K)$. First assume that $g\in[H\backslash G/K]-D(H,K)$. In this case $H\cap {}^gK<{}^gK$. Let

$$J_g={}^{g^{-1}}(H\cap {}^gK)={}^{g^{-1}}H\cap K<K.$$

Since

$$(M\uparrow_H^G)(J_g)=\bigoplus_{h\in[H\backslash G/J_g]} M(H\cap {}^hJ_g)$$

we conclude that $M(H\cap {}^gK)=M(H\cap {}^{g^{-1}}J_g)$ is the direct summand in $M(J_g)$ obtained for the double coset $Hg^{-1}J_g$ of $[H\backslash G/J_g]$). In particular, the map

$$\sum_{g\in[H\backslash G/K]-D(H,K)} (M\uparrow_H^G)(J_g)\longrightarrow (M\uparrow_H^G)(K) \tag{4.8}$$

is onto on the components $M(H\cap {}^gK)$ for $g\in[H\backslash G/K]-D(H,K)$. Suppose now that $g\in D(H,K)$ and that ${}^gK\leq H$ is not primordial for M. Let $L<{}^gK$. From formula (2.1) applied to ${}^{g^{-1}}L$, it follows that $M(L)$ is the direct summand in $(M\uparrow_H^G)({}^{g^{-1}}L)$ obtained for the double coset $Hg\,{}^{g^{-1}}L$. Since ${}^gK$ is not primordial for M, it follows that the map

$$\sum_{L<{}^gK}(M\uparrow_H^G)({}^{g^{-1}}L)\longrightarrow (M\uparrow_H^G)(K) \tag{4.9}$$

must be surjective on the $M({}^gK)$ component of $(M\uparrow_H^G)(K)$. Now if ${}^gK$ is not primordial for M, for any $g\in D(H,K)$, it follows from the surjectivity of maps (4.8) and (4.9) that the map

$$\sum_{L<K}(M\uparrow_H^G)(L)\longrightarrow (M\uparrow_H^G)(K)$$

is surjective as well, contradicting the fact that K is primordial for $M\uparrow_H^G$.

(2) Follows by analyzing the transfer maps (4.8) and (4.9) above. $\triangle$

The above proposition has the following corollary.

(4.1.10) COROLLARY.

(1) $\mathcal{P}(M_{G/H}) = \mathcal{P}(M) \cap \mathrm{Cl}_G(H)$.

(2) If $K \in \mathcal{P}(M_{G/H})$ *then*

$$\overline{M_{G/H}(K)} = \bigoplus_{g \in D(H,K)} \overline{M({}^gK)} \tag{4.10}$$

(3) If A *is a Green functor then* $\mathcal{D}(A_{G/H}) = \mathrm{SCl}_G(H) \cap \mathcal{D}(A)$.

PROOF. *(1)* From proposition (4.1.3) *(1)*, it follows that $\mathcal{P}(M\downarrow_H^G) = \mathcal{P}(M) \cap S(H)$. Now *(1)* follows from *(1)* of proposition (4.1.9) and from the fact that

$$M_{G/H} = (M\downarrow_H^G)\uparrow_H^G . \tag{4.11}$$

(2). From (4.1.3) *(1)* it follows that $\overline{(M\downarrow_H^G)(K)} = \overline{M(K)}$, for $K \in S(H)$. Since $\mathcal{P}(M)$ is closed under conjugation, it follows that if $K \in \mathcal{P}(M)$, and $[K] \leq [H]$, then ${}^gK \in \mathcal{P}(M\downarrow_H^G)$ for all $g \in D(H,K)$. Now *(2)* follows from formula (4.11) and *(2)* of proposition (4.1.9).

(3) Follows from *(1)* above, and from (4.1.7). $\triangle$

Let M and N be two Mackey functors. The following proposition gives a description of the Brauer quotients of $M \,\square\, N$.

(4.1.11) PROPOSITION.

(1) Let M *and* N *be two Mackey functors. Then*

$$\overline{(M \,\square\, N)(H)} = \overline{M(H)} \otimes \overline{N(H)}, \qquad \text{for all } H \in S(G). \tag{4.12}$$

In particular, $\mathcal{P}(M \,\square\, N) \subseteq \mathcal{P}(M) \cap \mathcal{P}(N)$.

(2) Let A *be a Green functor,* M *be a right* A *module, and* N *be a left* A *module. Then*

$$\overline{(M \,\square_A\, N)(H)} = \overline{M(H)} \otimes_{\overline{A(H)}} \overline{N(H)}, \qquad \text{for all } H \in S(G).$$

PROOF *(1)* Let $H \in S(G)$, and let $m \,\Box_H\, n \in (M \,\Box\, N)(H)$. If $m \in M(K)$ and $n \in N(K)$, for some $K < H$, then $m \,\Box_H\, n = t_K^H(m \,\Box_K\, n)$. Now assume that $m \in M(H)$, and $n \in N(H)$. If $m \in Tr_M(K)$, then let

$$m = \sum_{K<H} t_K^H(m_K), \qquad \text{for some } m_K \in M(K).$$

We conclude that

$$m \,\Box_H\, n = \sum_{K<H} \left(t_K^H(m_K)\right) \Box_H\, n = \sum_{K<H} m_K \,\Box_H \left(r_K^H(n)\right) =$$

$$\sum_{K<H} t_K^H\Big(m_K \,\Box_K \left(r_K^H(n)\right)\Big) \in Tr_{M \,\Box\, N}(H).$$

In conclusion, $\overline{(M \,\Box\, N)(H)}$ is generated by the elements $br_H^M(m) \,\Box_H\, br_H^N(n)$, for $m \in M(H)$ and $n \in N(H)$. Now *(1)* follows easily from relations (3.4)-(3.6).

(2) Follows from similar arguments. $\triangle$

Proposition (4.1.11) has the following immediate corollaries.

(4.1.12) COROLLARY.

(1) If $\mathcal{P}(M) \cap \mathcal{P}(N) = 0$, *then* $M \,\Box\, N = 0$.

(2) If A_1 *and* A_2 *are two Green functors, then* $D(A_1 \,\Box\, A_2) \subseteq D(A_1) \cap D(A_2)$.

(4.1.13) COROLLARY.

(1) Let M *be a left-A-module,* I *be a left ideal of* A *and* N *be a left submodule of* M. *If* $H \in S(G)$, *there exists a canonical epimorphism*

$$\overline{I(H)} \otimes_{\overline{A(H)}} \overline{N(H)} \longrightarrow \overline{(I \cdot N)(H)} \longrightarrow 0$$

In particular, $\mathcal{P}(I \cdot N) \subseteq \mathcal{P}(I) \cap \mathcal{P}(N)$.

(2) If $(I_i)_{i=1}^n$ *are left (two-sided) ideals of* A, *then* $\mathcal{P}\left(\prod_{i=1}^n I_i\right) \subseteq \bigcap_{i=1}^n \mathcal{P}(I_i)$.

(4.1.14) REMARK. In general $\mathcal{P}(I \cdot N)$ is not equal to $\mathcal{P}(I) \cap \mathcal{P}(N)$. To see this let $G = \mathbf{Z}_2$, and let A be the following Green functor:

$$A(H) = \begin{cases} \mathbf{Z}_4, & \text{if } H = G, \\ 0\ , & \text{if } H = 1, \end{cases}$$

and both r_1^G, t_1^G are zero. Let I be the ideal of A with $I(G) = 2A(G)$. Then $\mathcal{P}(I) = \{G\}$, but $I^2 = 0$.

(4.1.15) PROPOSITION.

Let $\mathcal{X} \subseteq S(G)$. Suppose that $(I_j)_{1\leq j\leq n}$ are ideals of A such that $\mathcal{P}(A/I_j) \subseteq \mathcal{X}$, for $j = 1, 2, ..., n$. Then:

(1)

$$\mathcal{P}\left(\frac{A}{\bigcap_{j=1}^{n} I_j}\right) \subseteq \mathcal{X}.$$

(2)

$$\mathcal{P}\Big(\frac{A}{\prod_{j=1}^{n} I_j}\Big) \subseteq \mathcal{X}.$$

In particular, if X is a finite G-set and A/I_j is X-projective, for $j = 1, 2, ..., n$, then both $A/\bigcap_{i=1}^{n} I_i$ and $A/\prod_{j=1}^{n} I_j$ are X-projective as well.

PROOF.*(1)* Let $n = 2$. Consider the exact sequence

$$0 \longrightarrow \frac{I_1}{I_1 \cap I_2} \longrightarrow \frac{A}{I_1 \cap I_2} \longrightarrow \frac{A}{I_1} \longrightarrow 0$$

Since

$$\frac{I_1}{I_1 \cap I_2} \cong \frac{I_1 + I_2}{I_2}$$

we conclude, from (4.1.3) *(2)*, that

$$\mathcal{P}\Big(\frac{I_1}{I_1 \cap I_2}\Big) \subseteq \mathcal{P}\Big(\frac{A}{I_2}\Big) \subseteq \mathcal{X}.$$

From the above exact sequence and (4.1.3) *(3)*, it follows that

$$\mathcal{P}\Big(\frac{A}{I_1 \cap I_2}\Big) \subseteq \mathcal{P}\Big(\frac{I_1}{I_1 \cap I_2}\Big) \bigcup \mathcal{P}\Big(\frac{A}{I_1}\Big) \subseteq \mathcal{X}.$$

The general case follows by induction over **n**.

(2) Again let $n = 2$. Consider the exact sequence

$$0 \longrightarrow \frac{I_2}{I_1 \cdot I_2} \longrightarrow \frac{A}{I_1 \cdot I_2} \longrightarrow \frac{A}{I_2} \longrightarrow 0$$

It is clear that $I_2/(I_1 \cdot I_2)$ is a left-A/I_1-module. From (4.1.3) *(2)*, we conclude that

$$\mathcal{P}\Big(\frac{I_2}{I_1 \cdot I_2}\Big) \subseteq \mathcal{P}\Big(\frac{A}{I_1}\Big) \subseteq \mathcal{X}.$$

From the above exact sequence it follows that

$$\mathcal{P}\Big(\frac{A}{I_1 \cdot I_2}\Big) \subseteq \mathcal{P}\Big(\frac{I_2}{I_1 \cdot I_2}\Big) \bigcup \mathcal{P}\Big(\frac{A}{I_2}\Big) \subseteq \mathcal{X}.$$

The general case follows by induction over n.

Now assume that $X = \coprod_{H \in \mathcal{X}} G/H$. From (4.3.7) *(1)*, it follows that a Green functor A' is X-projective if and only if $\mathcal{P}(A') \subseteq \mathrm{SCl}(\mathcal{X})$. From *(1)* and *(2)*, it follows immediately that both $A/\bigcap_{j=1}^{n} I_j$ and $A/\prod_{i=1}^{n} I_i$ are X-projective if the A/I_j are X-projective for $j = 1, 2, \ldots, n$. Δ

Proposition (4.1.15) has the following immediate corollary.

(4.1.16) COROLLARY.

Suppose that A is a Green functor, I is an ideal of A, and X is a finite G-set. If A/I is X-projective, then A/I^n is X-projective for all $n > 1$.

We conclude with a functorial characterization of the primordial subgroups of a Mackey functor.

Let $H \in S(G)$, and let $X_H = \coprod_{K<H} G/K$. The coproduct of the G-maps

$$\pi_K^H : G/K \longrightarrow G/H, \qquad \text{for } K < H,$$

is the G-map

$$\pi_H = \coprod_{K<H} \pi_K^H : X_H \longrightarrow G/H.$$

This G-map induces a morphism

$$\pi_H^* : M_{X_H} \longrightarrow M_{G/H}. \tag{4.12}$$

We have the following result.

(4.1.17) PROPOSITION.

$H \in \mathcal{P}(M)$, if and only if the map π_H^* given by the formula (4.12) *is not surjective.*

PROOF. Assume that $H \in \mathcal{P}(M)$. Since H is primordial, and $M(H) = M_{G/H}(G/G)$ we conclude that the map

$$\pi_H^*(G) : \bigoplus_{K<H} M(K) = M_{X_H}(G/G) = M(X_H \times G/G) \longrightarrow M_{G/H}(G/G) = M(H),$$

cannot be surjective.

Conversely, assume that $H \notin \mathcal{P}(M)$. In particular, $H \notin \mathcal{P}(M_{G/H})$. Notice that, if $K < H$, then every orbit of the G-set $G/H \times G/K$ is also an orbit of the G-set $X_H \times G/K$. In particular, this shows that the map

$$\pi_H^*(K) : M_{X_H}(G/K) = M(X_H \times G/K) \longrightarrow M(G/H \times G/K) = M_{G/H}(G/K)$$

is onto at K, for all $K < H$. Since, in this case, $\mathcal{P}(M_{G/H})$ consists of subgroups $[K] < [H]$ (thanks to (4.1.10)) it follows, from (4.1.5), that π_H is surjective. $\triangle$

4.2. THE TWIN FUNCTOR AND THE J FUNCTOR.

Let M be a G-Mackey functor over R and let $H \in S(G)$. If $K \leq H$, we let

$$br_K^H = br_K \cdot r_K^H : M(H) \longrightarrow \overline{M(K)}. \tag{4.13}$$

The map br_K^H is called *the Brauer homomorphism from H to K.* Consider the sum of all Brauer homomorphisms

$$\beta_M(H) = \bigoplus_{K \in \mathcal{P}(M) \cap S(H)} br_K^H : M(H) \longrightarrow \bigoplus_{K \in \mathcal{P}(M) \cap S(H)} \overline{M(K)}. \tag{4.14}$$

We write $\beta(H) = \beta_M(H)$ when the context is clear. Notice that H acts on

$$\bigoplus_{K \in \mathcal{P}(M) \cap S(H)} \overline{M(K)}$$

by acting simultaneously on the inedxing set and on the summands (see [T1]). Since H acts trivially on $M(H)$, it is easy to see that the image of $\beta_M(H)$ is contained in the fixed points

$$TM(H) = \left(\bigoplus_{K \in \mathcal{P}(M) \cap S(H)} \overline{A(K)} \right)^H. \tag{4.15}$$

When M is a Green functor for G over R, $TM(H)$ is an R-algebra with the product defined componentwise. In this case, $\beta(H)$ is a ring homomorphism because r_K^H and br_K are both ring homomorphisms.

In [T1] it is shown that the family of R-modules $TM(H)$, where $H \in S(G)$, inherits a natural G-Mackey functor structure. This is called the *twin functor TM of M.* When M is a Green functor TM is a Green functor as well. Moreover, if M is a Mackey (Green) functor then β_M is a map of Mackey (Green) functors.

The following two results appear in [T1].

(4.2.1) PROPOSITION ([T1]).

(1) If A is a Green functor, then $\mathcal{P}(A) = \mathcal{P}(TA)$.

(2) If A is a Green functor, then Ker β_A *is a nilpotent ideal of A.*

(3) If M is a Mackey functor and $H \in S(G)$, then both Ker $\beta_M(H)$ *and* Coker $\beta_M(H)$ *are annihilated by $n \cdot 1_R$, where n is the integer*

$$n = \prod_{K \in \mathcal{P}(M) \cap S(H)} |W_H K|. \tag{4.16}$$

(4.2.2) COROLLARY ([T1]).

(1) Let M be a left-A-module. If $|G|$ is invertible in R (or $A(G)$), then β_M is an isomorphism.

(2) If the rings $A(H)$ are commutative and reduced (i.e. $Nil(H) = 0$, for all $H \in S(G)$), then β_A is injective.

(4.2.3) DEFINITION. A Mackey functor M such that β_M is an isomorphism is called *totally decomposable.*

The totally decomposable Green functors are investigated in chapter 7. We conclude this chapter with an alternative description of the twin functor TM. We first need to introduce some notation and terminology.

(4.2.4) DEFINITION ([T3]) If $N \triangleleft G$ and $M' \in Mack_R(G/N)$, we let $\mathrm{Inf}^G_{G/N}(M')$ denote the G-Mackey functor defined by

$$\mathrm{Inf}^G_{G/N}(M')(H) = \begin{cases} M'(H/N), & \text{if } H \geq N, \\ 0 \quad , & \text{if } H \not\geq N \end{cases}$$

with obvious restriction, transfer and conjugation maps. We call $\mathrm{Inf}^G_{G/N}(M')$ the *inflation* of M'.

It is easy to see that $\mathrm{Inf}^G_{G/N}(M')$ is right adjoint to the *deflation* functor $\mathrm{Def}^G_{G/N}$ mapping a G-Mackey functor M to the G/N-Mackey functor $\mathrm{Def}^G_{G/N}(M)$ defined by

$$\mathrm{Def}^G_{G/N}(M)(H/N) = \frac{M(H)}{\sum_{K \not\geq N,\ K \leq H} t^H_K(M(K))}, \qquad \text{for all } H/N \in S(G/N). \tag{4.17}$$

If M is a G-Mackey functor, let $[G \backslash \mathcal{P}(M)]$ denote a set of representatives of $[\mathcal{P}(M)]_G$. The following result appears in [T3].

(4.2.5) PROPOSITION ([T3]).

Let M be a G-Mackey functor over R. For each $K \in \mathcal{P}(M)$, let $FP_{\overline{M(K)}}$ be the fixed point W_GK-Mackey functor associated with the $R[W_GK]$-module $\overline{M(K)}$ (see example (1.1.2)). Then

$$TM \cong \bigoplus_{K\in[G\backslash\mathcal{P}(M)]} \left(\mathrm{Inf}_{W_GK}^{N_GK} FP_{\overline{M(K)}}\right)\Big\uparrow_{N_GK}^{G}. \tag{4.18}$$

We now introduce the J-functor (see [Le1], [Le3], [Le4]).

(4.2.6) DEFINITION. Let $H \in S(G)$. There exists an obvious functor

$$Mack_R(G) \longrightarrow R[W_GH] - Mod \qquad M \longmapsto \overline{M(H)}. \tag{4.19}$$

The functor

$$J_{G/H} : R[W_GH] - Mod \longrightarrow Mack_R(G)$$

is the right adjoint of the functor given by (4.19). Let

$$j_{G/H}^{M} : M \longrightarrow J_{G/H}(\overline{M(H)})$$

be the unit of this adjunction. When the context is clear we write $j_{G/H}$ instead of $j_{G/H}^{M}$.

The following description of $J_{G/H}$ is due to Lewis (see [Le3], [Le4]).

(4.2.7) PROPOSITION ([Le3], [Le4]).

(1) Let $K \in S(G)$ and let $\overline{M}$ be an $R[W_GK]$-module. Then

$$J_{G/K}(\overline{M}) = \left(\mathrm{Inf}_{W_GK}^{N_GK} FP_{\overline{M}}\right)\Big\uparrow_{N_GK}^{G}. \tag{4.20}$$

.

It follows easily, by formula (4.20), that if $\overline{M}$ is an $R[W_GK]$-module then

$$\left(J_{G/K}(\overline{M})\right)(K) = \overline{M}. \tag{4.21}$$

As a consequence of (4.2.5) and (4.2.7) we have the following result.

(4.2.8) THEOREM ([Le3], [Le4]).

Let M be a G-Mackey functor (over R). Then

$$TM \cong \bigoplus_{K \in [G \backslash \mathcal{P}(M)]} J_{G/K}(\overline{M(K)}). \tag{4.22}$$

Moreover

$$\beta = \bigoplus_{K \in [G \backslash \mathcal{P}(M)]} j_{G/H}. \tag{4.23}$$

The main properties of the functor J are given in chapter 6. For additional information see [Le1], [Le3], [Le4].

5. Products of Elements in Green Functors.

Let A be a Green functor and M be a left-A-module. In this chapter we use approach 1.3 to define the external product of an element of A with an element of M. We use the external product to define concepts such as: submodules generated by one element, annihilators of submodules, units of Green functors, externally nilpotent elements of Green functors and zero divisors. We then give a description of these notions in terms of definition 1.1. As a corollary, we show that, if $(N(K))_{K\in\mathcal{P}(A)}$ is a collection of R-modules such that each $N(K)$ is a submodule of $M(K)$, and the family $(N(K))_{K\in\mathcal{P}(A)}$ is invariant under restrictions, transfers and conjugations, then this family can be uniquely extended to a submodule of M. We conclude with a few remarks about indecomposable units of classical Green functors. Throughout most of this chapter, we work with definition 1.3.

Let X, Y be finite G-sets. Using Yoneda lemma, one can think of elements $a \in A(X)$ and $m \in M(Y)$ as maps

$$B_X \xrightarrow{a} A, \qquad B_Y \xrightarrow{m} M.$$

Moreover, the composite

$$B_{X\times Y} \cong B_X \,\square\, B_Y \xrightarrow{a \,\square\, m} A \,\square\, M \xrightarrow{\zeta} M, \tag{5.1}$$

tells us that $\zeta(a \,\square\, m)$ can be regarded as an element of $M(X \times Y)$. Denote this element by

$$a \times b = \zeta(a \,\square\, m) \in M(X \times Y). \tag{5.2}$$

The operation $\times$ is called the *external product*. For an alternative description of $a \times m$, let

$$\pi_1 : X \times Y \longrightarrow X \qquad \pi_2 : X \times Y \longrightarrow Y$$

be the canonical projections. Then

$$a \times m = \pi_1^*(a) \cdot \pi_2^*(m) \in M(X \times Y). \tag{5.3}$$

For $a \in A(X)$ and $n > 0$, we denote by $[a]^n$ the n-th fold external product $a \times \ldots \times a \in A(X^n)$.

The following definition is due to Lewis:

(5.1) DEFINITION ([Le1]). *(1)* If $H \in S(G)$, then the elements $a \in A(H) = A(G/H)$ are called *indecomposable.*

(2) The *submodule generated by* $m \in M(X)$ is the image of the map

$$A \,\square\, B_X \xrightarrow{1_A \square m} A \,\square\, M \xrightarrow{\zeta} M,$$

and is denoted $A\langle m\rangle$.

(3) The *two-sided ideal generated by* $a \in A(X)$ is the image of the map

$$A \,\square\, B_X \,\square\, A \xrightarrow{1_A \square a \square 1_A} A \,\square\, A \,\square\, A \xrightarrow{\phi} A,$$

and is denoted $A\langle a\rangle A$. The right ideal generated by a is denoted $\langle a\rangle A$.

(4) An element $m \in M(X)$ is a *generator* of M if $A\langle m\rangle = M$.

(5) An element $u \in A(X)$ is a right (left) *unit* if u is a generator for A as a left (right) module over itself. An element $u \in A(X)$ is a *unit* (or a two-sided *unit*) if u is both a right and a left *unit*.

(6) An element $a \in A(X)$ *annihilates* M if $a \times m = 0$, for all $m \in M$. The set of all these elements is a two-sided ideal of A called the *annihilator of* M and denoted $\mathrm{Ann}_A(M)$.

(7) An element $a \in A(X)$ is a *zero divisor* if there exits $Y \in$ *G-Set* and a nonzero $b \in A(Y)$, such that $a \times b = 0$ or $b \times a = 0$. The element a is sometimes be called a Y-*zero divisor*.

(8) An element $a \in A(X)$ is *externally nilpotent* if $[a]^n = 0$ for some $n > 0$.

The following result is due to Lewis.

(5.2) PROPOSITION ([Le1]).

Let A be a Green functor. Then :

(1) A product of two right (left or two-sided) units is a right (left or two-sided) unit.

(2) If $u \in A(X)$ and $u \times v \in A(X \times Y)$ are right units with $v \in A(Y)$, then v is a right unit.

(3) A unit is not a zero divisor.

(4) $A(X)$ contains a one-sided unit if and only if the map $\theta_X : A_X \longrightarrow A$ is onto.

(5) Let $a \in A(X)$ and let $f : X \to Y$ be a map in C. If $f(a)$ is a right (left or two-sided) unit, then a is a right (left or two-sided) unit.

(6) An element $a \in A(X)$ is an X-<u>zero divisor</u> if and only if it is a zero divisor in the ring $A(X)$.

(7) Suppose A is a commutative Green functor. Then, an element $a \in A(X)$ is <u>externally nilpotent</u> if and only if it is internally nilpotent; that is nilpotent as an element of the ring $A(X)$.

From *(4)* of proposition (5.2) and theorem (2.5), we conclude:

(5.3) COROLLARY ([Le1]).

Let X be a G-set. Then A is X-projective if and only if A_X contains a one-sided <u>unit</u>.

Concerning nilpotency, we have the following theorem:

(5.4) THEOREM ([T4]).

Let A be a commutative Green functor. For $H \in S(G)$, let $Nil(H)$ be the nilradical of $A(H)$. Then the family $Nil(H)$, for $H \in S(G)$, is a Mackey functor which is an ideal of A.

From now on, the nilradical of a commutative Green functor A is denoted $Nil(A)$.

Again let A be a commutative Green functor, I be an ideal of A, and X be a finite G-set. Let

$$\sqrt{I}(X) = \{a \in A(X) \mid [a]^n \in I(X^n) \text{ for some } n > 0\}. \tag{5.4}$$

(5.5) THEOREM.

$\sqrt{I}$ is an ideal of A. Moreover, if $H \in S(G)$ then $\sqrt{I}(H)$ is the radical of the ideal $I(H)$.

PROOF. We apply (5.2) *(7)* to the Green functor A/I to conclude that, for $H \in S(G)$, $\sqrt{I}(H)$ is the radical of $I(H)$. The functorialy of $\sqrt{I}$ follows from theorem (5.4). $\triangle$

Let now X, Y be finite G-sets and $a \in A(X)$, $m \in M(Y)$. We describe $a \times m$, $A\langle m\rangle$ and $A\langle a\rangle A$ in terms of definition 1.1. Assume that $Y = \coprod_{H \in \mathcal{Y}} G/H$. Since

$$m \in M(Y) = M(\coprod_{H \in \mathcal{Y}} G/H) = \bigoplus_{H \in \mathcal{Y}} M(H)$$

it follows that the element $m \in M(Y)$ can be regarded as a set of elements $(m_H)_{H \in \mathcal{Y}}$ with $m_H \in M(H)$. Since

$$a \times m = \sum_{H \in \mathcal{Y}} a \times m_H$$

and

$$A\langle m\rangle = \sum_{H\in\mathcal{Y}} A\langle m_H\rangle,$$

it follows that it is enough to describe $a \times m$, $A\langle m\rangle$ and $A\langle a\rangle A$ when both a and m are indecomposable. Hence assume that $X = G/H$ and $Y = G/K$.

(5.6) PROPOSITION.

Let $a \in A(H)$ and $m \in M(K)$. Then

$$a \times m = \sum_{g\in[H\backslash G/K]} r^H_{H\cap {}^gK}(a) \cdot r^{{}^gK}_{H\cap {}^gK} c_g(m) \in M(G/H \times G/K) = \bigoplus_{g\in[H\backslash G/K]} M(H \cap {}^gK). \tag{5.5}$$

PROOF. Since

$$G/H \times G/K = \coprod_{g\in[H\backslash G/K]} G/H \cap {}^gK$$

formula (5.5) follows immediately from formula (5.3). $\triangle$

(5.7) PROPOSITION.

Let $H \in S(G)$, $a \in A(H)$, $m \in M(H)$, and $K \in S(G)$. Then

$$(A\langle m\rangle)(K) = \sum_{g\in[K\backslash G/H]} t^K_{K\cap {}^gH}\Big(A(K\cap {}^gH)\ r^{{}^gH}_{K\cap {}^gH} c_g(m)\Big), \tag{5.6}$$

$$(A\langle a\rangle A)(K) = \sum_{g\in[K\backslash G/H]} t^K_{K\cap {}^gH}\left(A(K\cap {}^gH)\ \Big(r^{{}^gH}_{K\cap {}^gH} c_g(m)\Big)\ A(K\cap {}^gH)\right). \tag{5.7}$$

PROOF. We check only formula (5.6). Denote the left-$A(K)$-submodule of $M(K)$ which appears in the right side of (5.6) by $N(K)$. The trick is to show that $\big(N(K)\big)_{K\in S(G)}$ is functorial because it is obvious that $N(K) \subseteq A\langle m\rangle(K)$, and $m \in N(H)$.

It is immediate that $c_g(N(K)) \subseteq N({}^gK)$. We now show that, if $L \subseteq K$, then $t^K_L N(L) \subseteq N(K)$. Fix

$$x = t^L_{L\cap {}^gH}(a r^{{}^gH}_{L\cap {}^gH}(c_g m)) \in N(L),$$

with $a \in A(L \cap {}^gH)$, and $g \in G$. Notice that $L \cap {}^gH \subset K \cap {}^gH$, hence

$$t^K_L(x) = t^K_L\Big(t^L_{L\cap {}^gH}(a r^{{}^gH}_{L\cap {}^gH}(c_g m))\Big) = t^K_{K\cap {}^gH}\left(t^{K\cap {}^gH}_{L\cap {}^gH}\Big(a r^{K\cap {}^gH}_{L\cap {}^gH}\big(r^{{}^gH}_{K\cap {}^gH}(c_g m)\big)\Big)\right) =$$

$$= t^K_{K\cap {}^gH}\Big(t^{K\cap {}^gH}_{L\cap {}^gH}(a) r^{{}^gH}_{K\cap {}^gH}(c_g m)\Big) \in N(K),$$

because $t_{L\cap {}^gH}^{K\cap {}^gH}(a) \in A(K\cap {}^gH)$. In the above argument, we have used *(i)* and the Frobenius axiom

(ix) from 1.1.

Finally let $J \subseteq L$. Then

$$r_J^L(x) = r_J^L\Big(t_{L\cap {}^gH}^L\big(ar_{L\cap {}^gH}^{{}^gH}(c_g m)\big)\Big) = \sum_{l\in[J\backslash L/L\cap {}^gH]} t_{J\cap {}^l(L\cap {}^gH)}^J r_{J\cap {}^l(L\cap {}^gH)}^{{}^l(L\cap {}^gH)} c_l\big(ar_{L\cap {}^gH}^{{}^gH}(c_g m)\big)$$

$$= \sum_{l\in[J\backslash L/L\cap {}^gH]} t_{J\cap {}^{lg}H}^J\Big(r_{J\cap {}^{lg}H}^{L\cap {}^{lg}H}(c_l(a))r_{J\cap {}^{lg}H}^{{}^{lg}H}(c_{lg}(m)\Big) \subseteq N(J),$$

because $r_{J\cap {}^{lg}H}^{L\cap {}^{lg}H}(c_l(a)) \in A(J\cap {}^{lg}H)$. In the above argument, we have used the fact that

$$J\cap {}^l(L\cap {}^gH) = J\cap {}^lL\cap {}^{lg}H = J\cap L\cap {}^{lg}H = J\cap {}^{lg}H,$$

and the Mackey axiom *(vi)* from 1.1. $\triangle$

Proposition (5.7) has the following corollaries.

(5.8) COROLLARY.

(1) If $m \in M(H)$, then $A\langle m\rangle(G) = t_H^G(A(H)m)$.

(2) If N_G is a left-$A(G)$-submodule of $M(G)$, then the value of the submodule

$$A\langle N_G\rangle = \sum_{n\in N_G} A\langle n\rangle$$

at G is exactly N_G.

(3) If $m \in M(H)$, and $P \in \mathcal{P}(A\langle m\rangle)$, then $[P] \leq [H]$.

Similar results hold for the principal two-sided ideals of A.

PROOF.

(1) Follows immediately from formula (5.6).

(2) Immediate consequence of *(1)*.

(3) One can obviously prove *(3)* using formula (5.6). For a simpler proof, notice that definition (5.1) *(2)* gives us an epimorphism

$$A_{G/H} \cong A \,\square\, B_{G/H} \xrightarrow{1_A \square m} A \,\square\, M \xrightarrow{\zeta} A\langle m\rangle.$$

Using proposition (4.1.3) and corollary (4.1.10), we conclude that

$$\mathcal{P}(A\langle m\rangle) \subseteq \mathcal{P}(A_{G/H}) \subseteq \mathcal{P}(A) \cap \mathrm{SCl}_G(H) \subseteq \mathrm{SCl}_G(H). \qquad \triangle$$

(5.9) COROLLARY (Submodules of a left-A-module).

Let M be a left-A-module. Suppose that for every $K \in \mathcal{P}(A)$, we are given an left-$A(K)$-submodule N_K of $M(K)$ such that:

(1) ${}^gN_K \subseteq N_{{}^gK}$, for $g \in G$.

(2) $r^L_K(N_L) \subseteq N(K)$, whenever $K \subseteq L$ in $\mathcal{P}(A)$.

(3) $t^L_K(N_K) \subseteq N(L)$, whenever $K \subseteq L$ in $\mathcal{P}(A)$.

Then there exists a unique an left-A-submodule N of M such that $N(K) = N_K$ for all $K \in \mathcal{P}(A)$.

PROOF. Let

$$N = \sum_{K \in \mathcal{P}(A)} \sum_{n \in N_K} A\langle n \rangle. \tag{5.8}$$

Using formula (5.6), it follows easily that $N(K) = N_K$ for all $K \in \mathcal{P}(A)$. If N_1 is another submodule of M with the asserted property, then, from formula (5.8), we conclude that $N \subseteq N_1$. On the other hand, $\mathcal{P}(N_1/N) \subseteq \mathcal{P}(A)$, and $(N_1/N)(K) = N_K/N_K = 0$ for all $K \in \mathcal{P}(A)$. In conclusion, $N_1/N = 0$. Hence $N = N_1$. Δ

(5.10) COROLLARY.

(1) Let N and N' be submodules of M. If $N(K) \subseteq N'(K)$ for all $K \in \mathcal{P}(A)$, then $N \subseteq N'$. If $N(K) = N'(K)$ for all $K \in \mathcal{P}(A)$, then $N = N'$.

(2) Suppose that A is a commutative Green functor. Let I be an ideal of A such that $I(K)$ is radical for all $K \in \mathcal{P}(A)$. Then $I = \sqrt{I}$.

(5.11) COROLLARY (Morphisms of left-A-modules).

Let M_1, M_2 be two left A modules. Suppose that for every $K \in \mathcal{P}(A)$, we are given a morphism $\theta_K : M_1(K) \longrightarrow M_2(K)$ of left-$A(K)$-modules such that:

(1) $c_g\theta_K = \theta_{{}^gK}c_g$, for $g \in G$.

(2) $r^L_K(\theta_L) = \theta_K(r^L_K)$, whenever $K \subseteq L$ in $\mathcal{P}(A)$.

(3) $t^L_K(\theta_K) = \theta_L(t^L_K)$, whenever $K \subseteq L$ in $\mathcal{P}(A)$.

Then there exists a unique morphism θ of A modules such that $\theta(K) = \theta_K$ for all $K \in \mathcal{P}(A)$.

PROOF. Assume that $H \notin \mathcal{P}(A)$, and let $m \in M(H)$. From formula (4.4), we conclude that

$$m = \sum_{K \in \mathcal{P}(A) \cap \mathcal{S}(H)} t_K^H(m_K), \qquad \text{for some } m_K \in M(K).$$

Define

$$\theta_H(m) = \sum_{K \in \mathcal{P}(A) \cap \mathcal{S}(H)} t_K^H(\theta_K(m_K)). \tag{5.9}$$

Using proposition (4.1.6) *(3)*, the axioms of definition 1.1, and induction on the subgroups $H \notin \mathcal{P}(A)$, one can check that $\theta = (\theta_H)_{H \in \mathcal{S}(G)}$ is well defined, and that it is a morphism of left A modules. The fact that θ is unique follows from formula (5.9). $\triangle$

(5.12) THEOREM.

Let X and Y be G-sets, and let $a \in A(X)$, and $m \in A(Y)$. Then:

$$A\langle a\rangle \cdot A\langle m\rangle = A\langle a \times m\rangle. \tag{5.10}$$

PROOF. Let ψ be the composite:

$$(A \,\square\, B_X) \,\square\, (A \,\square\, B_Y) \cong (A \,\square\, A) \,\square\, (B_X \,\square\, B_Y) \cong (A \,\square\, A) \,\square\, B_{X \times Y} \xrightarrow{\phi \,\square\, 1} A \,\square\, B_{X \times Y},$$

where the unlabeled isomorphisms above are the ones given by lemma (1.3.2). Let ϕ_a, ζ_m be the maps from definition (5.1), i.e.

$$A \,\square\, B_X \xrightarrow{1_A \,\square\, a} A \,\square\, A \xrightarrow{\phi} A\langle a\rangle, \qquad \phi_a = \phi \circ (1_A \,\square\, a),$$

$$A \,\square\, B_X \xrightarrow{1_A \,\square\, m} A \,\square\, M \xrightarrow{\zeta} A\langle m\rangle, \qquad \zeta_m = \zeta \circ (1_A \,\square\, m).$$

Let i_a, i_m be the inclusions

$$A\langle a\rangle \xrightarrow{i_a} A, \qquad A\langle m\rangle \xrightarrow{i_m} M,$$

of $A\langle a\rangle$ in A and $A\langle m\rangle$ in M, respectively.

Formula (5.10) now follows from the commutativity of the diagram

$$\begin{tikzcd}
(A \,\square\, B_X) \,\square\, (A \,\square\, B_Y) \arrow[r, "\phi_a \,\square\, \zeta_m"] \arrow[dr, "\psi"] & A\langle a\rangle \,\square\, A\langle m\rangle \arrow[r, "i_a \,\square\, i_m"] & A \,\square\, M \arrow[r, "\zeta"] & M \\
& A \,\square\, B_{X \times Y} \arrow[r, "1_A \,\square\, (a \times m)"] & A \,\square\, M \arrow[u, "\cong"'] &
\end{tikzcd}$$

$\triangle$

We conclude with a discussion about <u>units</u>. Notice that, if $D(A)$ is the defect set of A and $D = \coprod_{H \in D(A)} G/H$, then, according to proposition (5.2) and theorem (2.5), the ring

$A(D)$ contains units. We investigate the conditions under which A has indecomposable units. Using corollary (5.8), one concludes immediately that $u \in A(G)$ is a left (two-sided) unit if and only if u is a left (two-sided) unit in the ring $A(G)$. These indecomposable units will be called *trivial*.

The following proposition gives a necessary and sufficient condition for the existence of non-trivial indecomposable units.

(5.13) PROPOSITION.

$A(H)$ contains a unit if and only if $[K] \leq [H]$, for all $K \in \mathcal{P}(A)$.

PROOF. If $A(H)$ contains a unit, then, from corollary (5.8), it follows that $[K] \leq [H]$ for all $K \in \mathcal{P}(A)$. Conversely, assume that $[K] \leq [H]$ for all $K \in \mathcal{P}(A)$. From proposition (4.1.6) it follows that

$$1_{A(G)} = \sum_{K \in \mathcal{P}(A)} t_K^G(a_K), \qquad \text{for some } a_K \in A(K). \tag{5.11}$$

Let $K \in \mathcal{P}(A)$. Choose $g \in G$ such that ${}^gK \subseteq H$. Then $t_K^G(a_K) = t_{{}^gK}^G(c_g(a_K))$. This argument combined with equation (5.11) shows that there exists a relation

$$1_{A(G)} = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^G(b_K), \qquad \text{for some } b_K \in A(K)$$

Let $c_K = t_K^H(b_K)$. Then

$$1_{A(G)} = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_H^G(c_K) = t_H^G\Big(\sum_{K \in \mathcal{P}(A) \cap S(H)} c_K\Big).$$

If we let

$$c = \sum_{K \in \mathcal{P}(A) \cap S(H)} c_K,$$

then $c \in A(H)$, and c is a unit. Δ

We use (5.13) to conclude that some familiar Green functors have only trivial indecomposable units.

(5.14) EXAMPLE. Since G is primordial for the Burnside ring functor B (over any ring R), it follows that B has only trivial indecomposable units.

(5.15) EXAMPLE. Let $R_{\mathbf{C}}$ the character ring functor over $\mathbf{Z}$ of example (1.1.5). Assume that $H < G$, and that $u \in R_{\mathbf{C}}(H)$ is a unit. Then there exists $\phi \in R_{\mathbf{C}}(H)$ such that

$$1_{R_{\mathbf{C}}(G)} = t_H^G(\phi u).$$

By evaluating the above relation at e, the identity of G, we obtain

$$1 = |G/H|\phi(e)u(e) \equiv 0 \pmod{|G/H|},$$

because $\phi(e)u(e) = \dim(\phi u) \in \mathbf{Z}$. This relation shows that $G = H$, therefore this functor has no nontrivial indecomposable units.

(5.16) EXAMPLE. Let $A = R_{\mathbf{C}} \otimes \mathbf{Q}$ be the character ring functor over $\mathbf{Q}$ of example (1.1.5). Then (see [T4]), every cyclic subgroup of this Green functor is primordial. If $u \in A(H)$ is a nontrivial unit for this functor, we conclude that

$$\bigcup_{g \in G} {}^gH = G. \tag{5.12}$$

Indeed this relation follows because, for every element $x \in G$, the cyclic subgroup generated by x is primordial, hence it is contained in ${}^gH$ for some $g \in G$. However, notice that in equation (5.12), there are only $|G/N_G(H)|$ distinct sets ${}^gH$, which have nonempty intersection (because $e \in {}^gH$ for all $g \in G$). Moreover, all these sets have the same number of elements, namely $|H|$. Equation (5.12) implies that $|G/N_G(H)| \cdot |H| > |G|$, or $|H| > |N_G(H)|$. This shows that $R_{\mathbf{C}} \otimes \mathbf{Q}$ has no nontrivial indecomposable units.

(5.17) EXAMPLE. In this example, we construct a group G and a Green functor A for G, such that A has nontrivial indecomposable units. Let p be a prime number, and let $G = \mathbf{Z}_p^*$ be the cyclic group of the units inside the field $\mathbf{Z}_p$. Let $\zeta = \exp(2\pi i/p)$ be a root of unity of order p. Denote by $\mathcal{O}$ the ring of integers inside $\mathbf{Q}(\zeta)$. It is known that, as a $\mathbf{Z}$-module, $\mathcal{O} = \sum_{i=1}^{p-1} \mathbf{Z}\zeta^i$. Since G acts on $\mathcal{O}$ by automorphisms (as the Galois group of the extension $\mathbf{Q} \subset \mathbf{Q}(\zeta)$), we can consider the Green functor given by $A = FP_{\mathcal{O}}$ (see example (1.1.2)). Then $\zeta \in \mathcal{O} = A(1)$ is a unit for A. Indeed notice that

$$t_1^G(\zeta) = \sum_{i=1}^{p-1} \zeta^i = -1,$$

hence $A\langle\zeta\rangle(G) = A(G)$. It follows that $A\langle\zeta\rangle = A$. Similarly, let a and b be integers such that $-a + (p-1)b = \pm 1$. Then $a\zeta + b \in \mathcal{O} = A(1)$ is a unit for A.

6. Characteristic Mackey Functors.

Let $H \in S(G)$. In this chapter of we introduce the notion of an H-characteristic Mackey functor M. If M is H-characteristic for some $H \in S(G)$, we refer to M as being characteristic. The reason for investigating this concept is that some of the interesting Mackey functors such as simple Mackey functors, simple left-A-modules, prime and simple Green functors (see chapters 7 and 9 for definitions), turn out to be characteristic. The notion of a characteristic Mackey functor is due to Lewis (see [Le1]). From the results from [Le1] it follows easily that if M is H-characteristic then Min $\mathcal{P}(M) = [H]$. As it was pointed out in [Le1], the natural way of studying an H-characteristic Mackey functor M is via the natural map $j_{G/H} : M \longrightarrow J_{G/H}(M(H))$, where $J_{G/H}$ is the functor defined at (4.2.6). This is the approach that we adopt in our present work. If M is a Mackey functor and N is a submodule of M, then N is called cocharacteristic if M/N is characteristic. It turns out that some of the interesting subfunctors of a Mackey functor such as annihilators of characteristic left-A-modules, prime and maximal ideals, primary submodules of a left-A-module are cocharacteristic. We give a characterization theorem for the cocharacteristic submodules of a left-A-module. We show that every Mackey functor M has a canonical filtration

$$M = M_0 \supset M_1 \supset .... \supset M_n = 0$$

where M_{i+1} is cocharacteristic in M_i for $i = 1, ..., n$. If M is a left-A-module, then the above M_i are submodules of M. As a corollary of this result it follows that, if C_1 and C_2 are two Serre subclasses of the category of left-A-modules, then $C_1 \subseteq C_2$ if and only if every characteristic object in C_1 belongs to C_2. Due to this result, one can think of characteristic left-A-modules as analogs of cyclic modules from classical algebra. We also prove an induction theorem for Mackey functors satisfying a slightly weaker property than being characteristic. This form of our induction theorem is needed in chapters 12 and 13. As a corollary of this theorem we obtain some induction theorems for characteristic Mackey functors. The results of this chapter are crucial for most of the remainder of this work. The structure theorem for cocharacteristic subfunctors is used in an essential way throughout chapters 7, 8, 9, 10, 11, 12 and 13.

This chapter has two parts. In 6.1 we investigate the characteristic Mackey functors and the cocharacteristic subfunctors of a given Mackey functor M. In 6.2 we investigate the primordial subgroups of the H-characteristic G-Mackey functors (over R) in terms of H and of the prime numbers p dividing the order of G which are not invertible in R.

6.1. CHARACTERISTIC FUNCTORS AND COCHARACTERISTIC SUBMODULES.

(6.1.1) DEFINITION ([Le1]). Let H be a subgroup of G.

(1) A Mackey functor M is called *H-bounded* if $M(K) = 0$ for all $[K] < [H]$.

(2) A Mackey functor M is called *H-characteristic* if it is H-bounded, $M(H) \neq 0$, and the map

$$\theta^{G/H} : M \longrightarrow M_{G/H} \tag{6.1}$$

is injective. We refer to H as the characteristic subgroup of M. In what follows, we refer to the map (6.1) as θ_M or simply $M \longrightarrow M_{G/H}$. If M is H-characteristic for some $H \in S(G)$, we refer to M as being *characteristic*.

Notice that M is H-bounded if and only if H does not properly contain a primordial subgroup. The following proposition gives equivalent conditions for a Mackey functor to be characteristic.

(6.1.2) PROPOSITION.

Let $H \in S(G)$ and $M \in Mack_R(G)$. The following conditions are equivalent:

(1) M is H-characteristic.

(2) $\mathrm{Min}\ (\mathcal{P}(M)) = [H]$ *and θ_M is injective.*

(3) $\mathrm{Min}\ (\mathcal{P}(M)) = [H]$ *and for all $K \in S(G)$ we have* $\bigcap_{{}^{g}H \subseteq K} \mathrm{Ker}(r^K_{{}^{g}H}) = 0$.

PROOF. (1) $\Rightarrow$ (2). We show that if $K \in \mathcal{P}(M)$, then $[H] \leq [K]$. Assume that this is not the case. Let $K \in \mathcal{P}(M)$ be such that $[H] \not\leq [K]$. Then all stabilizers of orbits of the G-set $G/H \times G/K$ are strictly subconjugate to H. Since M is H-bounded it follows that $M(G/H \times G/K) = 0$. From the injectivity of map

$$\theta_M(G/K) : M(G/K) \longrightarrow M(G/H \times G/K)$$

it follows that $M(K) = M(G/K) = 0$. This contradicts the fact that $K \in \mathcal{P}(M)$.

(2) $\Rightarrow$ (1) Obvious.

(2) $\Leftrightarrow$ (3). Let $K \in S(G)$. Since Min $(\mathcal{P}(M)) = [H]$ it follows that, for $g \in G$, $M(K \cap {}^gH) = 0$ unless ${}^gH \subseteq K$. Hence

$$M(G/H \times G/K) = M\Big(\coprod_{g \in [K \backslash G/H]} G/(K \cap {}^gH)\Big) = \bigoplus_{g \in D(K,H)} M(G/{}^gH).$$

The map $\theta_M(K)$ is the map

$$\theta_M(K) = \bigoplus_{g \in D(K,H)} r^K_{{}^gH} : M(K) \longrightarrow \bigoplus_{g \in D(K,H)} M({}^gH).$$

It is clear that $\theta_M(K)$ is injective if and only if $\bigcap_{{}^gH \subseteq K} \mathrm{Ker}(r^K_{{}^gH}) = 0$. $\triangle$

(6.1.3) PROPOSITION.

(1) Let M be an H-characteristic Mackey functor and let N be a non-zero subfunctor of M. Then N is H-characteristic as well.

(2) If $(M_i)_{i \in \Gamma}$ are H-characteristic Mackey functors, then so is $\bigoplus_{i \in \Gamma} M_i$.

(3) If M is an H-characteristic Mackey functor and $X \in G$-Set, then M_X is either H-characteristic or zero.

PROOF. *(1)* It is clear that N is H-bounded. Moreover, since the natural diagram

$$\begin{array}{ccc} N & \xrightarrow{\theta_N} & N_{G/H} \\ \downarrow n & & \downarrow n_{G/H} \\ M & \xrightarrow{\theta_M} & M_{G/H} \end{array}$$

commutes, it follows that θ_N is injective.

(2) Let $M = \bigoplus_{i \in \Gamma} M_i$. It follows immediately that M is H-bounded and $M(H) \neq 0$. Moreover, since the maps

$$M_i \longrightarrow (M_i)_{G/H},$$

are injective, so is their direct sum

$$M \longrightarrow \bigoplus_{i \in \Gamma} (M_i)_{G/H} = \Big(\bigoplus_{i \in \Gamma} M_i\Big)_{G/H} = M_{G/H}.$$

(3) By *(2)*, it is enough to assume that $X = G/K$. If $[H] \not\leq [K]$, it follows immediately that $M_{G/K} = 0$. Assume now that $H \leq K$. Since $\mathcal{P}(M_{G/K}) \subseteq \mathcal{P}(M)$ (by corollary (4.1.10)), and M is H-bounded, we conclude that $M_{G/K}$ is H-bounded. Since $H \leq K$,

it follows that $M(H)$ appears as a direct summand in $M_{G/K}(H)$; hence $M_{G/K}(H) \neq 0$. Finally from the injectivity of the map

$$M \longrightarrow M_{G/H}$$

we conclude that the map

$$M_{G/K} \longrightarrow \left(M_{G/H}\right)_{G/K} \cong \left(M_{G/K}\right)_{G/H}$$

is injective as well. Δ

Let $H \in S(G)$. Let $H\text{-}Mack_R(G)$ be the full subcategory of $Mack_R(G)$ consisting of the H-bounded Mackey functors. Notice that if $M \in H\text{-}Mack_R(G)$, then $\overline{M(H)} = M(H)$. In [Le1] it is shown that if $M \in H\text{-}Mack_R(G)$, then the map $d^0 = \theta_M$ of the standard complex (2.2) for $X = G/H$,

$$0 \longrightarrow M \xrightarrow{d^0} M_{G/H} \xrightarrow{d^1} M_{G/H \times G/H} \xrightarrow{d^2} \ldots$$

has a factorization

$$\begin{tikzcd} M \arrow[rr, "d^0"] \arrow[dr, "j_{G/H}"'] & & M_{G/H} \\ & J_{G/H}(M(H)) \arrow[ur, "i"'] & \end{tikzcd}$$

where $J_{G/H}$ is the functor defined at (4.2.6). Here the map i is the canonical inclusion $\operatorname{Ker} d^1 \overset{i}{\hookrightarrow} M_{G/H}$. The following three results are due to Lewis.

(6.1.4) PROPOSITION ([Le1]).

Let $M \in H\text{-}Mack_R(G)$. Then:

(1) M is H-characteristic if and only if $j_{G/H}$ is injective.

(2) M satisfies G/H-injective induction if and only if $j_{G/H}$ is an isomorphism.

A Mackey functor $M \in H\text{-}Mack_R(G)$ satisfying condition *(2)* above is called *H-determined.* Proposition (6.1.4) *(2)* can be rephrased as follows.

(6.1.5) PROPOSITION ([Le1]).

The correspondence

$$\overline{M} \longmapsto J_{G/H}(\overline{M})$$

is a natural equivalence between the category of $R[W_G H]$-modules and the category of H-determined Mackey functors.

(6.1.6) PROPOSITION ([Le1]).

Let M, $N \in H\text{-}Mack_R(G)$, and let $\overline{X}$ be an $R[W_GH]$-module. Then there is a one-to-one correspondence between maps

$$M \,\square\, N \longrightarrow J_{G/H}(\overline{X})$$

and $R[W_GH]$-maps

$$M(H) \otimes N(H) \longrightarrow \overline{X}$$

where W_GH acts diagonally on $M(H) \otimes N(H)$.

We introduce the following definition.

(6.1.7) DEFINTION. A subfunctor N of a Mackey functor M is *H-cocharacteristic* if M/N is H-characteristic. We sometimes refer to M as an *H-cocharacteristic extension* of M. We refer to H as the cocharacteristic subgroup of N. If N is H-cocharacteristic for some $H \in S(G)$ we refer to N as being *cocharacteristic.*

Our next important result is theorem (6.1.14) which describes all the cocharacteristic subfunctors of a given Mackey functor M. We begin by introducing the following notion.

(6.1.8) DEFINITION. Let S be a ring with a finite group G acting on it. A left S-module N is called G-equivariant if G acts on N in such a way that the actions of G on S and N are compatible. By the compatibility condition we mean that if $g \in G$, $s \in S$, $m \in N$, and if we let $c_g(s)$ and $c_g(m)$ be the actions of g on s and m respectively, then $c_g(s \cdot m) = c_g(s) \cdot c_g(m)$. Notice that if N is G-equivariant, then a submodule of N is G-equivariant if and only if it is G-invariant. Let *G-S-Mod* be the category of all G-equivariant left-S-modules. For more information on *G-S-Mod* see Appendix A1.

(6.1.9) EXAMPLE. Let A be a Green functor and let M be a left-A-module. If $H \in S(G)$, then, by axiom *(viii)* of definition (1.1), we conclude that $M(H)$ is an W_GH-equivariant left-$A(H)$-module. If $H \in \mathcal{P}(M)$, then $\overline{M(H)}$ is an W_GH-equivariant left-$\overline{A(H)}$-module.

Let A be a Green functor. We give a property of the annihilator of an H-bounded left-A-module M (for the definition of the annihilator of M see (5.1) *(6)*).

(6.1.10) LEMMA.

Let A be a Green functor and let M be an H-bounded left-A-module. Then $A/\mathrm{Ann}_A(M)$ is H-bounded. In particular, the ideal $\mathrm{Ann}_A(M)$ contains the ideal

$$I = \sum_{K<H} A\langle 1_{A(K)} \rangle. \tag{6.2}$$

PROOF. Let $K < H$. We show that $1_{A(K)} \in (\mathrm{Ann}_A(M))(K)$. Let $X \in G\text{-}Set$ and let $m \in M(X)$. Notice that all stabilizers of orbits of the G-set $(G/K) \times X$ are strictly subconjugate to H. Since M is H-bounded it follows that $M((G/K) \times X) = 0$. Since $1_{A(K)} \times m \in M((G/K) \times X)$ it follows that $1_{A(K)} \times m = 0$.

The fact that $\mathrm{Ann}_A(M)$ contains I follows by noticing that I is the smallest ideal J of A such that A/J is H-bounded. $\triangle$

We are now ready to prove the following theorem.

(6.1.11) THEOREM.

(1) Let $\overline{X}$ be an $R[W_GH]$-algebra. Then $J_{G/H}(\overline{X})$ is a Green functor.

(2) Let A be a Green functor and let $H \in \mathcal{P}(A)$. Suppose that $\overline{X}$ is an W_GH-equivariant left-$\overline{A(H)}$-module. Then $J_{G/H}(\overline{X})$ has a natural left-A-module structure.

(3) Let A be a Green functor and let $H \in \mathcal{P}(A)$. If $\overline{X}$ is a ring which is an W_GH-equivariant $\overline{A(H)}$-algebra then $J_{G/H}(\overline{X})$ is a Green functor and has a natural A-algebra structure.

PROOF. *(1)* See [Le1] proposition (5.10).

(2) Let I be the ideal given by formula (6.2). Then A/I is H-bounded. Moreover, by formula (5.6),

$$I(H) = \sum_{K<H} t_K^H(A(K)) = Tr_A(H);$$

hence $(A/I)(H) = \overline{A(H)}$. By proposition (6.1.6) and formula (4.21), the natural $R[W_GH]$-map

$$\overline{A(H)} \otimes \overline{X} \longrightarrow \overline{X}$$

corresponds to a map

$$(A/I) \,\square\, J_{G/H}(\overline{X}) \longrightarrow J_{G/H}(\overline{X}).$$

Hence $J_{G/H}(\overline{X})$ is a left-A/I-module, therefore a left-A-module.

(3) Immediate consequence of *(1)* and *(2)*. Δ

Theorem (6.1.11) can be rephrased as follows.

(6.1.12) THEOREM.

(1) Let $H \in S(G)$. The functor $J_{G/H}$ is right adjoint to the evaluation functor

$$Green_R(G) \longrightarrow R[W_GH]\text{-}Alg \qquad A \longrightarrow \overline{A(H)}. \tag{6.3}$$

(2) Let A be a Green functor and let $H \in \mathcal{P}(A)$. The functor $J_{G/H}$ is right adjoint to the evaluation functor

$$A - Mod \longrightarrow W_GH\text{-}\overline{A(H)}\text{-}Mod \qquad M \longrightarrow \overline{M(H)}. \tag{6.4}$$

Let A be a Green functor and M be a left-A-module. For $H \in \mathcal{P}(A)$ the maps

$$j_{G/H} : A \longrightarrow J_{G/H}(\overline{A(H)}, \qquad j_{G/H}(M) \longrightarrow J_{G/H}(\overline{M(H)})$$

are the units of the adjunctions given by formulae (6.3) and (6.4), respectively. By theorem (6.1.12), we obtain the following result.

(6.1.13) COROLLARY.

Let A be a Green functor, $M \in A$-Mod and $H \in \mathcal{P}(M)$. Then the maps

$$j_{G/H} : A \longrightarrow J_{G/H}(\overline{A(H)}), \qquad j_{G/H} : M \longrightarrow J_{G/H}(\overline{M(H)})$$

are maps of Green functors and left-A-modules, respectively.

It follows, by theorem (4.2.8) and corollary (6.1.3), that if M is a left-A-module, then TM is a left-A-module. In this case, the canonical map $\beta_M : M \longrightarrow TM$ is a map of left-A-modules.

Now we are ready to describe the cocharacteristic submodules of a left-A-module M. Notice that if N is H-characteristic then, by propositions (6.1.2) and (4.1.3), it follows that $H \in \mathcal{P}(M/N) \subseteq \mathcal{P}(M)$.

(6.1.14) THE CONSTRUCTION OF $M_{(H,\ \overline{N_H})}$. Let A be a Green functor and let M be a left-A-module. Let $H \in \mathcal{P}(M)$ and let $\overline{N_H}$ be a proper W_GH-invariant submodule of $\overline{M(H)}$. Let n_H be the composite

$$M \xrightarrow{j_{G/H}} J_{G/H}(\overline{M(H)}) \xrightarrow{J_{G/H}(\overline{n_H})} J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right), \tag{6.5}$$

where $\overline{n_H}$ is the projection $\overline{M(H)} \longrightarrow \overline{M(H)}/\overline{N_H}$. It is clear that n_H is a map of left-A-modules. Since $M/(\mathrm{Ker}(n_H))$ can be regarded as a non-zero submodule of the H-characteristic left-A-module

$$J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right),$$

it follows, by proposition (6.1.3) *(1)*, that $M/(\mathrm{Ker}(n_H))$ is H-characteristic. Hence $\mathrm{Ker}(n_H)$ is H-cocharacteristic. Denote $\mathrm{Ker}(n_H)$ by $M_{(H,\, \overline{N_H})}$. The next theorem shows that all the H-cocharacteristic submodules of M are of the form $M_{(H,\, \overline{N_H})}$ for some proper W_GH-invariant submodule $\overline{N_H}$ of $\overline{M(H)}$.

(6.1.15) THEOREM (The Structure of Cocharacteristic Submodules of M).

Let A be a Green functor, M be a left-A-module and $H \in \mathcal{P}(M)$. There exists a one-to-one correspondence between the proper W_GH-invariant submodules of $\overline{M(H)}$ and the H-cocharacteristic submodules of M. This correspondence is obtained as follows. If N is H-cocharacteristic, then $N(H)$ is a proper W_GH-invariant submodule of $M(H)$ containing $Tr_M(H)$. If we set

$$\overline{N_H} = \frac{N(H)}{Tr_M(H)}$$

then

$$N = M_{(H,\, \overline{N_H})}. \tag{6.6}$$

PROOF. Assume that N is H-cocharacteristic. Since $(M/N)(H) \neq 0$, it follows that $N(H)$ is a proper W_GH-invariant submodule of $M(H)$. Since M/N is H-bounded, it follows that $(M/N)(K) = 0$ whenever $K < H$. Hence $N(K) = M(K)$ for $K < H$. From the functoriality of N it follows that

$$N(H) \supseteq \sum_{K<H} t_K^H(N(K)) = \sum_{K<H} t_K^H(M(K)) = Tr_M(H).$$

Let

$$\overline{N_H} = \frac{N(H)}{Tr_M(H)}.$$

It follows, by a fundamental isomorphism theorem, that

$$\left(\frac{M}{N}\right)(H) = \frac{M(H)}{N(H)} \cong \frac{\overline{M(H)}}{\overline{N_H}}. \tag{6.7}$$

It follows, by proposition (6.1.4) and formula (6.7), that the natural map

$$j_{G/H} : \frac{M}{N} \longrightarrow J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right)$$

is injective. Let $\pi_N : M \longrightarrow M/N$ be the canonical projection. From the commutative diagram

$$\begin{tikzcd} \frac{M}{N} \arrow[r, "j_{G/H}"] & J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right) \\ & M \arrow[ul, "\pi_N"] \arrow[u, "n_H"'] \end{tikzcd}$$

we conclude that $N = \mathrm{Ker}(n_H) = M_{(H,\, \overline{N_H})}$. $\triangle$

We now interpret the correspondence given by formula (6.6) in terms of definition (1.1). Let N_H be a proper W_GH-invariant submodule of $M(H)$ containing $Tr_M(H)$. Then there exists a unique H-cocharacteristic submodule N of M such that $N(H) = N_H$ (compare to corollary (5.9)). If $K \in S(G)$ then

$$N(K) = \begin{cases} M(K), & \text{for } [H] \not\leq [K], \\ {}^gN_H, & \text{for } K = {}^gH, \\ \bigcap\limits_{{}^gH \subseteq K} (r^K_{{}^gH})^{-1}\, {}^gN_H, & \text{for } [H] \leq [K]. \end{cases} \tag{6.8}$$

Formula (6.8) clearly shows that N is uniquely determined by its value at H. If we set $\overline{N_H} = N(H)/Tr_M(H)$, then $N = M_{(H,\, \overline{N_H})}$.

(6.1.16) OBSERVATION. Let A be a Green functor and let $H \in \mathcal{P}(A)$. If $\overline{I_H}$ is a proper left (two-sided) W_GH-invariant ideal of $\overline{A(H)}$, then $A_{(H,\, \overline{I_H})}$ is a left (respectively two-sided) H-cocharacteristic ideal of A.

(6.1.17) EXAMPLE. Let B be the Burnside ring Green functor for G (over any ring R) of example (1.1.6). Then (see [T4]) $\overline{B(H)} = R$, for all $H \in S(G)$. Moreover, W_GH acts trivially on $\overline{B(H)}$. Theorem (6.1.15) gives a one-to-one correspondence between the proper ideals I of R and the H-cocharacteristic ideals $B_{(H,\, I)}$ of B. When $R = \mathbb{Z}$, we obtain that every H-cocharacteristic ideal of B is of the form $B_{(H,\, n\mathbb{Z})}$ for some $n > 1$ or $n = 0$.

(6.1.18) REMARK. The operator $M_{(H,\, -)}$ preserves and reflects containments. In other words, if $\overline{N_1}$ and $\overline{N_2}$ are two proper W_GH-invariant submodules of $\overline{M(H)}$, then

$$M_{(H,\, \overline{N_1})} \subseteq M_{(H,\, \overline{N_2})}$$

if and only if $\overline{N_1} \subseteq \overline{N_2}$.

Let A be a Green functor and let X be a finite G-set. Lewis (see [Le1]) asked if one can determine all prime ideals of the Green functor A_X provided that the prime ideals of

A are known. In chapter 9 we show that the prime ideals of any Green functor A are of the form $A_{(H,\overline{I_H})}$ for some proper W_GH-invariant ideal of $\overline{A(H)}$ satisfying a certain technical condition. Due to this result it follows that it is enough to characterize the cocharacteristic ideals of the Green functor A_X. The next theorem gives an explicit description of the left (two-sided) ideals of $A_{G/K}$. Let $H \in \mathcal{P}(A_{G/K})$. From corollary (4.1.10), we know that $H \in \mathcal{P}(A) \cap \mathrm{SCl}_G(K)$. Hence we may assume $H \subseteq K$. Notice that W_GH acts on the right on $D(K, H)$ by

$$(KgH,\ wH) \longrightarrow KgwH.$$

It is straight-forward group theoretical calculation to show that

$$\mathrm{Stab}_{W_GH}(KgH) = \frac{{}^{g^{-1}}K \cap N_GH}{H} = W_{{}^{g^{-1}}K}H. \tag{6.9}$$

Let m be the number of orbits in $D(K, H)/W_GH$, and let Kg_1W_GH, ..., Kg_mW_GH be the orbits of $D(K, H)/W_GH$. With these notations we have:

(6.1.19) THEOREM.

There exists a one-to-one correspondence between the H-cocharacteristic left (two-sided) ideals of $A_{G/K}$ and the proper left (two-sided) ideals $(I_1,\ I_2,\ ...,\ I_m)$ of $\overline{A(H)}^m$ such that I_i is $W_{{}^{g_i^{-1}}K}H$-invariant for $i = 1,\ ...,\ m$.

PROOF. According to theorem (6.1.15), the left (two-sided) H-cocharacteristic ideals of $A_{G/K}$ are in one-to-one correspondence with the proper left (two-sided) W_GH-invariant ideals of $\overline{A_{G/K}(H)}$. From corollary (4.1.10), we know that

$$\overline{A_{G/K}(H)} = \prod_{g \in D(K,H)} \overline{A({}^gH)}. \tag{6.10}$$

Every left (two-sided) ideal of the ring on the right side of equation (6.10) is of the form

$$\prod_{g \in D(K,H)} I_{{}^gH},$$

for some $I_{{}^gH}$ left (two-sided) ideal of $\overline{A({}^gH)}$. Moreover, W_GH acts on the right of the ring on the right side of formula (6.10) by

$$\Big((a_{{}^gH})_{g \in D(K,H)},\ wH\Big) \longmapsto \big(c_{gwg^{-1}}(a_{{}^{gw}H})\big)_{gw \in D(K,H)}.$$

Hence the left (two-sided) ideals $I_{{}^gH}$ satisfy ${}^{gwg^{-1}}I_{{}^gH} = I_{{}^{gw}H}$ for all $w \in \mathrm{Stab}_{W_GH}(KgH)$. The assertion of the theorem follows easily from formula (6.9). $\triangle$

(6.1.20) COROLLARY.

Let A be a Green functor and $H \in \mathcal{P}(A)$. There is a one-to-one correspondence between the left (two-sided) H-cocharacteristic ideals of $A_{G/H}$ and the proper left (two-sided) ideals of $\overline{A(H)}$.

PROOF. Follows immediately from theorem (6.1.19) because $D(H,H)/W_GH$ has only one element, namely HeH, and its stabilizer is $W_HH = \{e\}$ (e being the identity element of the group G). Δ

Next we prove some miscellaneous properties of the cocharacteristic submodules. We show that the operator $M_{(H,\,-)}$ commutes with the intersections in the following sense:

(6.1.21) PROPOSITION.

Let $(\overline{N_i})_{i\in\Gamma}$ be a family of proper W_GH equivariant left-$\overline{A(H)}$-submodules of $\overline{M(H)}$, and let $\overline{N}$ be their intersection. Then

$$M_{(H,\,\overline{N})} = \bigcap_{i\in\Gamma} M_{(H,\,\overline{N_i})}.$$

PROOF. For simplicity, let $\mathcal{N}_i = M_{(H,\,\overline{N_i})}$, for $i \in \Gamma$, and let $\mathcal{N} = M_{(H,\,\overline{N})}$. Notice that there exists a canonical injection

$$\frac{M}{\bigcap_{i\in\Gamma}\mathcal{N}_i} \longrightarrow \bigoplus_{i\in\Gamma}\frac{M}{\mathcal{N}_i}.$$

From proposition (6.1.3) *(1)* and *(2)*, we conclude that $\bigcap_{i\in\Gamma}\mathcal{N}_i$ is H-cocharacteristic. Since

$$\frac{\left(\bigcap_{i\in\Gamma}\mathcal{N}_i\right)(H)}{T_M(H)} = \bigcap_{i\in\Gamma}\overline{N_i} = \overline{N} = \frac{\mathcal{N}(H)}{T_M(H)},$$

it follows that

$$\Big(\bigcap_{i\in\Gamma}\mathcal{N}_i\Big)(H) = \mathcal{N}(H).$$

From formula (6.8) we conclude that $\mathcal{N} = \bigcap_{i\in\Gamma}\mathcal{N}_i$. Δ

(6.1.22) THEOREM (The Cocharacteristic Decomposition Theorem).

Let N be a proper submodule of M. Then N is an intersection of cocharacteristic submodules of M if and only if

$$\beta_{M/N} : M/N \longrightarrow T(M/N)$$

is injective.

PROOF. By replacing M with M/N we can assume that $N = 0$. Notice that if $H \in \mathcal{P}(M)$, then $M_{(H,\,0)}$ is the unique minimal H-cocharacteristic submodule of M. Moreover

$$M_{(H,\,0)} = \operatorname{Ker} n_H, \quad \text{where} \quad n_H : M \longrightarrow J_{G/H}(\overline{M(H)}).$$

Notice also that if $[H] = [K]$ then $M_{(H,\,0)} = M_{(K,\,0)}$. Hence,

$$\bigcap_{H \in \mathcal{P}(M)} M_{(H,\,0)} = \bigcap_{H \in [G \backslash \mathcal{P}(M)]} M_{(H,\,0)} = \bigcap_{H \in [G \backslash \mathcal{P}(M)]} \operatorname{Ker} n_H = 0$$

if and only if the map

$$\sum_{H \in [G \backslash \mathcal{P}(M)]} n_H : M \longrightarrow \bigoplus_{H \in [G \backslash \mathcal{P}(M)]} J_{G/H}(\overline{M(H)}) = TM$$

is injective. But this map is obviously the map β_M. $\triangle$

We give an example of a Mackey functor M and a subfunctor N of M such that N is not an intersection of cocharacteristic subfunctors in M.

(6.1.23) EXAMPLE. Let G be a finite group, and let p be a prime number such that $p \mid |G|$. Let B be the Burnside ring Green functor for G over $\mathbf{Z}$. Let $J = pB$. We show that J is not an intersection of cocharacteristic ideals in B. Notice first that B/J can be identified with B_p, the Burnside ring functor for G over $\mathbf{Z}_p$. Since $\overline{B_p(H)} = \mathbf{Z}_p$, one can use the formula (4.15) to conclude that the Green functor TB_p has no nilpotent elements. However, B_p has nilpotent elements. It follows that the map

$$\beta_{B_p} : B_p \longrightarrow TB_p$$

cannot be injective. From theorem (6.1.21), we conclude that J cannot be an intersection of cocharacteristic ideals.

There is one case in which every submodule of a module M is an intersection of cocharacteristic submodules.

(6.1.24) EXAMPLE. Let M be a non-zero left-A-module, and assume that $|G|$ is invertible in R (or in $A(G)$). Then every submodule of M is an intersection of cocharacteristic submodules in M. Indeed, let N be a submodule of M. From corollary (4.2.2), we conclude that the map

$$\beta_{M/N} : M/N \longrightarrow T(M/N)$$

is an isomorphism. Now the claim about N follows from theorem (6.1.22).

(6.1.25) THEOREM (The Transitivity of Cocharacteristic Extensions).

Let $N_1 \subset N_2 \subset N_3$. If N_i is H-cocharacteristic in N_{i+1}, for $i = 1, 2$, then N_1 is H-cocharacteristic in N_3.

PROOF. Since $N_1(K) = N_2(K) = N_3(K)$, for $[K] < [H]$, we conclude that $N_3/N_1 \in H\text{-}Mack_R(G)$. Moreover, since $N_1(H) \neq N_2(H) \neq N_3(H)$, it follows that $(N_3/N_1)(H) \neq 0$. Now the result follows from the fact that, in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N_2/N_1 \arrow[r] \arrow[d, "\theta_{12}"'] & N_3/N_1 \arrow[r] \arrow[d, "\theta_{13}"'] & N_3/N_2 \arrow[r] \arrow[d, "\theta_{23}"] & 0 \\
0 \arrow[r] & (N_2/N_1)_{G/H} \arrow[r] & (N_3/N_1)_{G/H} \arrow[r] & (N_3/N_2)_{G/H} \arrow[r] & 0,
\end{tikzcd}$$

the maps θ_{12} and θ_{23} are both injective. Hence θ_{13} is injective as well. $\triangle$

(6.1.26) PROPOSITION.

Let M be a Mackey functor and let N be H-cocharacteristic in M.

(1) Let X be a finite G-set. If $N_X \neq M_X$ then N_X is H-cocharacteristic in M.

(2) If $N \subset N_1 \subseteq M$ then N is H-cocharacteristic in N_1.

PROOF. Follows by proposition (6.1.3) applied to the characteristic functor M/N. $\triangle$

The following notions are used in chapters 7 and 8.

(6.1.27) DEFINITION. Let M be a Mackey functor. Let

$$\mathcal{P}_1(M) = \text{Min } (\mathcal{P}(M)),$$

$$\mathcal{P}_i(M) = \text{Min } \Big(\mathcal{P}(M) - \big(\bigcup_{j<i} \mathcal{P}_j\big)\Big), \qquad \text{for } i > 1. \tag{6.11}$$

Let $H \in \mathcal{P}(M)$. The integer i such that $H \in \mathcal{P}_i(M)$ is called the *height of H in M*, and is denoted by $i = \text{ht}_M(H)$. Since G is finite we conclude that $\mathcal{P}_m(M) = \emptyset$ for some $m > 1$.

The smallest integer m with this property is denoted pl_M, and is called the *primordial length of M.* Notice that $pl_M = \max\,(\mathrm{ht}_M(H) \mid H \in \mathcal{P}(M))$. If M is a left-A-module, then $pl_M \leq pl_A$, and $\mathrm{ht}_M(H) \leq \mathrm{ht}_A(H)$ for all $H \in \mathcal{P}(M)$.

(6.1.28) THEOREM.

Let M be a Mackey functor. Then M has a finite filtration

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0 \tag{6.12}$$

such that M_{i+1} is cocharacteristic in M_i for $i = 1, \ldots, n-1$. If A is a Green functor and M is a left-A-module, then the subfunctors M_i can be chosen to be submodules of M.

PROOF. Let

$$l = \max\,\big(|H| \mid H \in \mathcal{P}(A)\big)$$

and

$$k_M = \min\,\big(|K| \mid \mathrm{ht}_M(K) = 1\big).$$

Notice that $M(K) = 0$ if $|K| < k_M$. Moreover, notice that if $K \in S(G)$ is such that $M(K) \neq 0$ but $[K] \neq [K_i]$ for $i = 1, \ldots, s$, then $|K| > k_M$. We proceed by induction on $l - k_M$. Let $K_1, \ldots, K_s$ be representants for the conjugacy classes of subgroups in $\mathrm{Min}\,\big(\mathcal{P}(M)\big)$. For $1 \leq i \leq s$ let

$$N_i = M_{(K_i,\,0)}$$

and

$$M_i = \bigcap_{1 \leq j \leq i} N_j. \tag{6.13}$$

Notice that $N_i(K_i) = 0$ for $i = 1, \ldots, s$. Let $i \in \{1, \ldots, s-1\}$. We show that $M_{i+1} \subset M_i$. Indeed, otherwise $M_{i+1} = M_i$; hence

$$\bigcap_{1 \leq j \leq i} N_j \subseteq N_{i+1}.$$

It follows, by evaluating the above subfunctors at K_{i+1}, that

$$\bigcap_{1 \leq j \leq i} N_j(K_{i+1}) \subseteq N_{i+1}(K_{i+1}). \tag{6.14}$$

However, since K_j are representants for some conjugacy classes of minimal primordial subgroups of M, it follows that $[K_j] \not\leq [K_{i+1}]$ for $j = 1, \ldots, i$. Since N_j is K_j-cocharacteristic, it follows that $N_j(K_{i+1}) = M(K_{i+1}) \neq 0$ for $1 \leq j \leq i$. Since $N_{i+1}(K_{i+1}) = 0$, the containment (6.14) becomes

$$M(K_{i+1}) = 0.$$

This contradiction shows that $M_{i+1} \subset M_i$. We now show that M_{i+1} is cocharacteristic in M_i. It follows, by a fundamental theorem of isomorphism, that

$$\frac{M_i}{M_{i+1}} = \frac{M_i}{M_i \cap N_{i+1}} \cong \frac{M_i + N_{i+1}}{N_{i+1}}. \tag{6.15}$$

Since N_{i+1} is cocharacteristic in M it follows, by proposition (6.1.26) *(2)* and formula (6.15), that M_{i+1} is cocharacteristic in M_i. Hence

$$M = M_0 \supset M_1 \supset \dots \supset M_s \tag{6.16}$$

is a descending chain of subfunctors of M such that M_{i+1} is cocharacteristic in M_i for $i = 1, \dots, s-1$. Now let $N = M_s$. If $N = 0$ we are done. If not let $K \in \mathrm{Min}\,(\mathcal{P}(N))$. Since $N \subseteq N_i$ and $N_i(K_i) = 0$, it follows that $[K] \neq [K_i]$ for $i = 1, \dots, s$. Since $M(K) \supseteq N(K) \neq 0$, it follows that $|K| > k_M$. From this argument it follows that

$$k_N = \min\left(|K| \mid \mathrm{ht}_N(K) = 1\right) > k_M;$$

hence $m - k_N < m - k_M$. By the induction hypothesis, N has a filtration

$$M_s = N = N_0 \supset N_1 \supset \dots \supset N_t = 0 \tag{6.17}$$

such that N_{i+1} is cocharacteristic in N_i for $i = 1, \dots, t-1$. The assertion of the theorem follows from formulae (6.16) and (6.17). If M is a left-A-module, then the subfunctors M_i given by formula (6.13) are submodules of M. Δ.

(6.1.29) OBSERVATION. The length n of the filtration in formula (6.12) is bounded by a constant which depends only on the group G.

(6.1.30) DEFINITION.

Let A be a Green functor and let C_1 be a subcategory of A-*Mod*.

(WSC) C_1 is called a *WSC* (weak Serre class) if C_1 satisfies the following properties:

(WSC-1) C_1 is a full subcategory of A-*Mod*.

(WSC-2) C_1 is closed under finite direct sums.

(WSC-3) If $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ is a short exact sequence in A-*Mod*, with $M \in C_1$, then $N \in C_1$ and $L \in C_1$.

(SC) C_1 is called a SC (Serre class) if C_1 satisfies the following properties:

(SC-1) Same as (WSC-1).

(SC-2) If $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ is a short exact sequence in A-Mod, then $M \in C_1$ if and only if both $N \in C_1$ and $L \in C_1$.

Notice that every SC is a WSC.

The following corollary is an immediate consequence of theorem (6.1.28).

(6.1.31) COROLLARY.

Let C_1 and C_2 be two subcategories of A-Mod such that C_1 is a WSC and C_2 is a SC. Then $C_1 \subseteq C_2$ if and only if every characteristic left-A-module in C_1 is contained in C_2.

6.2. INDUCTION THEOREMS.

Let A be a Green functor and let M be a left-A-module. We begin with some properties of $\mathrm{Ann}_A(M)$. Notice that if $H \in \mathcal{P}(M)$, then $\mathrm{Ann}_{A(H)}(\overline{M(H)}) \supset Tr_A(H)$. It follows easily that

$$\frac{\mathrm{Ann}_{A(H)}(\overline{M(H)})}{Tr_A(H)} = \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)}). \tag{6.18}$$

(6.2.1) PROPOSITION.

Let A be a Green functor, M be a left-A-module and $H \in S(G)$.

(1) $\mathrm{Min}\ \mathcal{P}(M) = \mathrm{Min}\ \mathcal{P}(A/\mathrm{Ann}_A(M))$.

(2) If the natural map

$$M \longrightarrow M_{G/H}$$

is injective, then the natural map

$$\frac{A}{\mathrm{Ann}_A(M)} \longrightarrow \Big(\frac{A}{\mathrm{Ann}_A(M)}\Big)_{G/H}$$

is injective.

PROOF. Set $A_1 = A/\mathrm{Ann}_A(M)$. It is clear that M is a left-A_1-module. Moreover, one can easily show that $\mathrm{Ann}_{A_1}(M) = 0$.

(1) It is immediate that $\mathrm{Min}\ \mathcal{P}(M) \subseteq \mathcal{P}(M) \subseteq \mathcal{P}(A_1)$. We show that $\mathrm{Min}\ \mathcal{P}(A_1) \subseteq \mathcal{P}(M)$. Suppose that this is not the case, and let $H \in \mathrm{Min}\ \mathcal{P}(A_1) - \mathcal{P}(M)$. Since $M(K) = 0$, for all $[K] \leq [H]$, we conclude that $M_{G/H} = 0$. Now the external multiplication with $1_{A_1(H)}$ induces the map θ_M

$$M \longrightarrow M_{G/H}, \qquad m \longmapsto 1_{A_1(H)} \times m.$$

Since $M_{G/H} = 0$, we conclude that $1_{A_1(H)} \in \mathrm{Ann}_{A_1}(M)(H) = 0$. Hence $A_1(H) = 0$. But this contradicts the fact that $H \in \mathcal{P}(A_1)$.

(2) Let θ' be the composite

$$A \longrightarrow A_1 \longrightarrow (A_1)_{G/H}.$$

It is enough to show that $a \in \mathrm{Ker}\ \theta'$ implies that $a \in \mathrm{Ann}_A(M)$. It is easy to see that $\theta_A(a) = 1_{A(H)} \times a$ and $\theta_M(m) = 1_{A(H) \times m}$, where θ_A and θ_M are the maps given by formula (6.1). Notice that $1_{A(H)}$ commutes with the external multiplication up to isomorphisms. Thus,

$$\begin{aligned} a \in \mathrm{Ker}\ \theta' \ &\Leftrightarrow\ 1_{A(H)} \times a \in \mathrm{Ann}_A(M) \ \Leftrightarrow\ \forall m \in M,\ 1_{A(H)} \times a \times m = 0 \\ &\Leftrightarrow\ \forall m \in M,\ a \times 1_{A(H)} \times m = 0 \ \Leftrightarrow\ \forall m \in M,\ 0 = a \times \theta_M(m) = \theta_M(a \times m) \\ &\Leftrightarrow\ \forall m \in M,\ a \times m = 0. \end{aligned}$$

Here, the last step follows because θ_M is an injective A-module homomorphism. $\triangle$

(6.2.2) COROLLARY.

Let A be a Green functor and let M be an left-A-module.

(1) If M is H-characteristic and N is a non-zero submodule of M, then $\mathrm{Ann}_A(N)$ *is H-cocharacteristic.*

(2) If $\mathrm{Ann}_A(N)$ *is H-cocharacteristic whenever N is a nonzero submodule of M, then M is H-characteristic.*

(3) If M is an H-characteristic left-A-module, then

$$\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{Ann}_{\overline{A(H)}} M(H))}. \tag{6.19}$$

PROOF. *(1)* From (6.1.3) *(1)*, it follows that it is enough to assume that $M = N$. We apply proposition (6.2.1) to conclude that $[H] = \mathrm{Min}\ \mathcal{P}(M) = \mathrm{Min}\ \mathcal{P}(A/\mathrm{Ann}_A(M))$, and that

$$\frac{A}{\mathrm{Ann}_A(M)} \longrightarrow \Big(\frac{A}{\mathrm{Ann}_A(M)}\Big)_{G/H}$$

is injective. From proposition (6.1.2), we conclude that $\mathrm{Ann}_A(M)$ is H-cocharacteristic.

(2). Let N be a non-zero submodule of M. Let $K \in \mathrm{Min}\ (\mathcal{P}(N))$. Choose a nonzero $m \in N(H)$, and let $N_1 = A\langle m\rangle$. Then $[K] = \mathrm{Min}\ \mathcal{P}(N_1) = \mathrm{Min}\ \mathcal{P}(A/\mathrm{Ann}_A(N_1)) = [H]$. In particular, $[H] = \mathrm{Min}\ \mathcal{P}(M)$. Let θ_M be the map

$$\theta_M : M \longrightarrow M_{G/H}.$$

Notice that Ker $\theta_M(K) = 0$, for all $[K] \leq [H]$. It follows that if θ_M were not injective, then H would not be a primordial subgroup of the non-zero submodule Ker θ_M. But this would contradict the previous argument. Hence M is H-characteristic.

(3) We show that $\mathrm{Ann}_{A(H)}(M(H)) = \mathrm{Ann}_A(M)(H)$. It is enough to show that $\mathrm{Ann}_{A(H)}M(H) \subseteq \mathrm{Ann}_A(M)(H)$. Indeed, if this is not the case, let

$$I = A\langle \mathrm{Ann}_{A(H)}(M(H))\rangle = \sum_{a \in \mathrm{Ann}_{A(H)}(M(H))} A\langle a\rangle$$

Since $I \not\subseteq \mathrm{Ann}_A(M)$, we conclude that $I \cdot M \neq 0$. Since $I \cdot M$ is H-characteristic it follows that $(I \cdot M)(H) \neq 0$. Since $[H] = \mathrm{Min}\ \mathcal{P}(M)$ and $\mathrm{Ann}_{A(H)}(M(H)) \supset Tr_A(H)$, we use corollary (3.1.4) *(1)* to conclude that $(I \cdot M)(H) = I(H) \cdot M(H) = 0$. This contradicts the fact that $I \cdot M$ is H-characteristic.

Now formula (6.19) follows from *(1)* and formula (6.18). $\triangle$

Notice that if M is a left-A-module, then $\mathrm{Ann}_A(M)(H) \subseteq \mathrm{Ann}_{A(H)}(M(H))$. However, they are not equal in general, as is shown by the following example.

(6.2.3) EXAMPLE. Let p be a prime number, and let G be a nontrivial p-group. Let S be the trivial $\mathbf{Z}[G]$-algebra $\mathbf{Z}_p$. Let $A = FP_S$. Now let

$$M(H) = \begin{cases} 0\ , & \text{if } H \neq 1, \\ \mathbf{Z}_p, & \text{if } H = 1. \end{cases}$$

It is clear that M is an A-module. Since M is 1-characteristic and $\mathrm{Ann}_{A(1)}M(1) = 0$, it follows, by formula (6.19), that $\mathrm{Ann}_A(M) = A_{(1,\ 0)}$. Since A is 1-characteristic and $\mathrm{Ann}_A(M)(1) = 0$, we conclude that $\mathrm{Ann}_A(M) = 0$. On the other hand, for $H \neq 1$, $\mathrm{Ann}_{A(H)}(M(H)) = \mathbf{Z}_p$.

We now prove the following induction theorem.

(6.2.4) THEOREM.

Let p be a prime number and let $H \in S(G)$. Let A be a Green functor such that $q \cdot 1_{A(G)}$ is invertible in $A(G)$ whenever q is a prime number not equal to p which divides $|G|$.

(1) Assume that M is a left-A-module which satisfies the following conditions:

(i) The map $M \longrightarrow M_{G/H}$ is injective.

(ii) $[H_p] \leq [K]$, for all $K \in \mathcal{P}(M)$.

Then all the primordial subgroups of M satisfy $[H_p] \leq [K] \leq [H^p]$. Moreover, M is G/H^p-projective.

(2) If M is an H-characteristic left-A-module, then M is G/H^p-projective. Moreover, if $K \in \mathcal{P}(M)$, then $[H] \leq [K] \leq [H^p]$.

PROOF. *(1)* Let $A_1 = A/\mathrm{Ann}_A(M)$. From proposition (6.2.1) it follows that A_1 satisfies *(i)* and *(ii)*. Moreover, since M is a left-A_1-module, it follows, by proposition (4.1.3), that $\mathcal{P}(M) \subseteq \mathcal{P}(A_1)$. From theorem (2.6), it follows that if X is a finite G-set such that A_1 is X-projective, then M is X-projective as well. Hence, it is enough to prove the theorem when M is replaced with A_1. Replace also A with A_1.

Let $K \in \mathcal{P}(A)$. Let

$$\mathcal{X} = \{K \cap {}^gH \mid g \in G \text{ such that } [H_p] \leq [K \cap {}^gH]\ \}. \tag{6.20}$$

Notice that $\mathcal{X}$ is closed under conjugation with elements from K. From the injectivity of the map

$$A(K) \longrightarrow A_{G/H}(K) = \bigoplus_{g \in [K\backslash G/H]} A(K \cap {}^gH),$$

we conclude that

$$\bigcap_{g \in [K\backslash G/H]} \mathrm{Ker}\ r^K_{K\cap {}^gH} = 0. \tag{6.21}$$

Since $A(K \cap {}^gH) = 0$ if $K \cap {}^gH \notin \mathcal{X}$, we conclude that

$$\bigcap_{L \in \mathcal{X}} \mathrm{Ker}\ r^K_L = 0. \tag{6.22}$$

Apply theorem (2.9) to the Green functor $A{\downarrow}^G_K$ to conclude that

$$\sum_{L \in \mathcal{H}^K_p \mathcal{X}} t^K_L A(L) + \bigcap_{L \in \mathcal{X}} \mathrm{Ker}\ r^K_L = |K|^{\perp}_p A(K). \tag{6.23}$$

We use formula (6.22), and the fact that $|K|^{\perp}_p \cdot 1_{A(K)}$ is invertible in $A(K)$, to conclude that

$$\sum_{L \in \mathcal{H}^K_p \mathcal{X}} t^K_L A(L) = A(K). \tag{6.24}$$

Since K is primordial it follows that $K \in \mathcal{H}^K_p \mathcal{X}$; hence $[K_p] \in \mathrm{SCl}_K(\mathcal{X})$. It follows that $K_p \leq L$ for some $L \in \mathcal{X}$. From formula (6.20), it follows that $L = K \cap {}^gH$ for some $g \in G$ such that ${}^gH_p \leq L$. Since ${}^gH_p$ is normal in ${}^gH$, it follows that ${}^gH_p$ is normal in L. Since $K_p \leq L$, it follows that $K_p \cap {}^gH_p$ is normal in K_p. Then

$$\frac{K_p}{K_p \cap {}^gH_p} \cong \frac{{}^gH_p \cdot K_p}{{}^gH_p}. \tag{6.25}$$

Since ${}^{g}H_p \leq L \leq {}^{g}H$, it follows that $L/{}^{g}H_p$ is a p-group. From the isomorphism (6.25) it follows that

$$\frac{K_p}{K_p \cap {}^{g}H_p}$$

is a p-group as well. Since $(K_p)_p = K_p$, it follows that $K_p \cap {}^{g}H_p = K_p$; hence ${}^{g}H_p \leq K_p$. However, since

$${}^{g}H_p \leq K_p \leq L \leq {}^{g}H,$$

and since ${}^{g}H_p$ is normal in ${}^{g}H$, it follows that ${}^{g}H_p$ is normal in K_p. Now $K_p/{}^{g}H_p$ is a subgroup of the p-group ${}^{g}H/{}^{g}H_p$. Since $(K_p)_p = K_p$, it follows that ${}^{g}H_p = K_p$. Since $[H_p] = [K_p]$ it follows that $[K] \leq [H^p]$. In particular, A_1 satisfies G/H^p-projective induction.

(2) Obvious consequence of *(1)*. $\triangle$

We now investigate induction theory for arbitrary H-characteristic Mackey functors.

Let Π be a set of prime numbers dividing $|G|$. We set

$$\mathbf{Z}_{(\Pi)} = \left\{\frac{a}{b} \in \mathbf{Q} \mid \text{all the prime divisors of } b \text{ are in } \Pi\right\},$$

$$R_{(\Pi)} = R \otimes \mathbf{Z}_{(\Pi)}.$$

Let $B_{(\Pi)} = B \otimes R_{(\Pi)}$ be the Burnside ring functor for G over $R_{(\Pi)}$, and let $A_{(\Pi)} = A \,\square\, B_{(\Pi)}$.

(6.2.5) THEOREM (Induction Theorem for Characteristic Mackey Functors).

Let M be an H-characteristic left-A-module. Assume that Π is a set of prime numbers dividing $|G|$ (which may be empty), such that $q \cdot 1_{A(G)}$ is invertible in $A(G)$, whenever $q \mid |G|$ and $q \in \Pi'$. Then, for every $K \in \mathcal{P}(M)$, there exists $p \in \Pi$, such that

$$[H] \leq [K] \leq [H^p]. \tag{6.26}$$

If $K \in \mathcal{P}(M)$ is not conjugate to H, then $\overline{M(K)}$ is annihilated by the integer

$$n = \max\ (|W_K{}^{g}H| \mid {}^{g}H \subseteq K), \tag{6.27}$$

which is a power of p. Moreover, M is $\coprod_{p \in \Pi} G/H^p$-projective.

PROOF. Let $A_1 = A/\mathrm{Ann}_A(M)$. From proposition (6.2.2), it follows that $A/\mathrm{Ann}_A(M)$ is H-characteristic. Since M is a left-A_1-module, it follows that $\mathcal{P}(M) \subseteq \mathcal{P}(A_1)$. Moreover, it follows, by theorem (2.6), that if X is a finite G-set such that A_1 is X-projective, then M is X-projective as well. Hence, it is enough to prove the theorem when M is replaced by A_1. Replace also A with A_1.

According to theorem (6.2.4) *(2)*, we need only consider the case where Π has at least two elements. For each $p \in \Pi$, let $\Pi_{p'} = \Pi - \{p\}$. One can easily check that $A_{(\Pi_{p'})}$ is also H-characteristic, and that it satisfies the hypothesis of theorem (6.2.4) *(2)*. It follows that

$$[H] \leq [K] \leq [H^p], \qquad \text{for all } K \in \mathcal{P}(A_{(\Pi_{p'})}). \tag{6.28}$$

In particular, if $K \in S(G)$ is such that $[H] < [K] \not\leq [H^p]$, then there exists an integer $n_{(p)}$, whose prime divisors are in $\Pi - \{p\}$, such that $\overline{A(K)}$ is annihilated by $n_{(p)}$. Assume now that $K \in S(G)$, is such that $[H] < [K] \not\leq [H^p]$ for all $p \in \Pi$. Since the numbers $(n_{(p)})_{p\in\Pi}$ have no common divisor, and $n_{(p)}$ annihilates $\overline{A(K)}$, for all $p \in \Pi$, it follows that $\overline{A(K)} = 0$. Hence K is not primordial.

Suppose now that $K \in \mathcal{P}(A)$ is such that for some $p \in \Pi$, $[H] < [K] \leq [H^p]$. For $g \in D(K, H)$, let

$$\alpha_g = \log_p |W_K {}^gH|. \tag{6.29}$$

Notice that α_g are integers. Denote by $\alpha = \max\ (\alpha_g \mid g \in D(K,H))$, and notice that $n = p^\alpha$, where n is given by formula (6.26). Consider the element of $A(K)$ given by

$$x = p^\alpha \cdot 1_{A(K)} - \sum_{g\in D(K,H)} p^{\alpha-\alpha_g} \cdot t^K_{{}^gH}(1_{A({}^gH)}). \tag{6.30}$$

One can use the Mackey axiom and the fact that $[H] = \operatorname{Min} \mathcal{P}(A)$ to conclude that $r^K_{{}^gH}(x) = 0$ for all ${}^gH \subseteq K$. From proposition (6.1.2) *(3)* it follows that $x = 0$. From formula (6.30), it follows that $n \cdot 1_{A(K)} = p^\alpha \cdot 1_{A(K)} \in Tr_A(K)$. Hence n annihilates $\overline{A(K)}$. $\triangle$

7. The Jacobson Radical.

Throughout this chapter, A is a Green functor and M is a left-A-module. This chapter has four parts.

In 7.1 we define the simple left-A-modules. We give a characterization theorem for these modules and we investigate their annihilators. We also give an induction theorem for such modules. The proof of this induction theorem is based on some results from chapter 9.

In 7.2 we define the Jacobson radical $Jac(A)$ of A as the intersection of the annihilators of all simple left-A-modules. Unlike in classical algebra, this ideal does not coincide with the intersection of all maximal ideals of A. We give various descriptions of $Jac(A)$. From our results it follows that $Jac(A)$ coincides with the intersection of annihilators of all simple right-A-modules. This answers a question from [Le1].

In 7.3 we investigate the properties of $Jac(A)$. We show that finitely generated left-A-modules satisfy the Nakayama lemma. We also show that $Jac(A)$ contains any nilpotent ideal of A. If A is commutative, we show that $Jac(A)$ contains $Nil(A)$ (for the ideal $Nil(A)$, see theorem (5.4)). Let X be a finite G-set. We show that $Jac(A_X) \cong (Jac(A))_X$. We also show that $Jac(A)(X)$ consists of all elements $a \in A(X)$ such that $1_{A(X\times Y)} - a \times b$ is a unit in $A(X \times Y)$ for all $Y \in G\text{-}Set$ and all $b \in A(Y)$. For $H \in S(G)$, let $Jac(A(H))$ be the Jacobson radical of $A(H)$. We show that $\big(Jac(A)\big)(H) \subseteq Jac(A(H))$. We also show that $Jac(A)$ is the largest ideal J of A such that $J(H) \subseteq Jac(A(H))$ for all $H \in S(G)$.

In 7.4 we assume that A is commutative. We investigate the totally decomposable Green functors A. We show that A is totally decomposable if and only if every simple A-module M is projective relative to its primordial subgroup. We show that if A is totally decomposable then $Jac(A)$ is the intersection of all maximal ideals of A.

7.1. SIMPLE LEFT-A-MODULES.

(7.1.1) DEFINITION. Let A be a Green functor. A left-A-module M is *simple* if $M \neq 0$ and the only submodules of M are 0 and M.

(7.1.2) DEFINITION. Let S be a ring with a finite group G acting on it. Let M' be a G-equivariant left-S-module (see definitions (6.1.8) and (A1.3)). The module M' is *simple* G-equivariant if $M' \neq 0$ and the only G-invariant submodules of M' are 0 and M' (see also definition (A1.7)).

(7.1.3) THEOREM (Characterization Theorem for Simple Left-A-Modules).

A left-A-module M is simple if and only if M satisfies the following three conditions:

(1) $\mathcal{P}(M) = [H]$, for some $H \in \mathcal{P}(M)$.

(2) M is H-characteristic.

(3) $M(H)$ is a simple W_GH-equivariant left-$\overline{A(H)}$-module.

PROOF. Suppose that M is simple, and let $H \in \text{Min } \mathcal{P}(M)$. Both maps

$$\theta_{G/H} : M_{G/H} \longrightarrow M \tag{7.1}$$

and

$$\theta^{G/H} : M \longrightarrow M_{G/H} \tag{7.2}$$

are non-zero at H. From the simplicity of M we conclude that $\theta_{G/H}$ is surjective (hence M satisfies *(1)*) and $\theta^{G/H}$ is injective (hence M satisfies *(2)*). Finally, notice that $M(H) = \overline{M(H)}$. Using theorem (6.1.15) and the simplicity of M, we conclude that M satisfies *(3)*.

Conversely, assume that M satisfies the conditions *(1)*-*(3)* above. Let N be a non-zero submodule of M. Using proposition (6.1.3) *(1)* it follows that $N(H) \neq 0$. Since $N(H)$ is an W_GH-invariant submodule of $M(H)$, we conclude that $N(H) = M(H)$. Finally, since $\mathcal{P}(M/N) \subseteq \mathcal{P}(M) = [H]$, and $(M/N)(H) = 0$, we conclude that $\mathcal{P}(M/N) = \emptyset$. Hence $N = M$. $\quad\triangle$

(7.1.4) LEMMA.

Let A be a Green functor and let M be a simple left-A-module. Suppose that $\mathcal{P}(M) = [H]$. Then

$$\text{Ann}_A(M) = A_{(H,\ \text{Ann}_{\overline{A(H)}}(M(H)))}. \tag{7.3}$$

PROOF. Since M is H-characteristic, it follows, by corollary (6.2.2), that $\mathrm{Ann}_A(M)$ is H-cocharacteristic. Formula (7.3) follows from formula (6.19). $\triangle$

Let S be a ring with a finite group G acting on it. If I is an ideal of S let

$$\mathrm{eq}(I) = \bigcap_{g \in G} {}^{g}I. \tag{7.4}$$

(7.1.5) PROPOSITION (Annihilators of Simple Left-A-Modules).

Let A be a Green functor and let $H \in \mathcal{P}(A)$. Let $\overline{Jac_H}$ be the Jacobson radical of the ring $\overline{A(H)}$.

(1) Let $\overline{m_H}$ be a maximal ideal of $\overline{A(H)}$. Then there exists a simple left-A-module M such that

$$\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{eq}(\overline{m_H}))}. \tag{7.5}$$

(2) Let M be a simple left-A-module. Assume that M is H-characteristic. Then

$$A_{(H,\ \overline{Jac_H})} \subseteq \mathrm{Ann}_A(M). \tag{7.6}$$

(3) Suppose that A is commutative and let M be a simple A-module. Then

$$\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{eq}(\overline{m_H}))}$$

for some maximal ideal $\overline{m_H}$ of $\overline{A(H)}$.

PROOF. *(1)* We use theorem (A1.11) to conclude that there exists a W_GH-equivariant simple left-$\overline{A(H)}$-module M'_H such that

$$\mathrm{Ann}_{\overline{A(H)}}(M'_H) = \mathrm{eq}(\overline{m}). \tag{7.7}$$

Let

$$M' = J_{G/H}(M'_H).$$

Consider the following submodule of M':

$$M = \sum_{n \in M'(H)} A\langle n \rangle.$$

From formula (4.21) and proposition (5.6), we conclude that $M(H) = M'(H) = M'_H$. Since M' is H-characteristic, it follows that M is H-characteristic as well. We use corollary (5.8)

(3) to conclude that $\mathcal{P}(M) = [H]$. From theorem (7.1.3), we conclude that M is simple. Finally from formulae (7.3) and (7.7) it follows that

$$\mathrm{Ann}_A(M) = A_{\left(H,\ \mathrm{Ann}_{\overline{A(H)}}(M(H))\right)} = A_{(H,\ \mathrm{eq}(m))}.$$

(2) Let M be a simple left-A-module which is H-characteristic. Since $M(H)$ is simple W_GH-equivariant, it follows, by theorems (7.1.3) and (A1.11) *(3)*, that $\overline{Jac_H} \subseteq \mathrm{Ann}_{\overline{A(H)}}(M(H))$. Formula (7.6) follows from formula (7.3) and from the fact that the operator $A_{(H,\ -)}$ preserves containments.

(3) Immediate consequence of theorem (A1.11) *(4)*. $\triangle$

(7.1.6) DEFINITION. Let A be a Green functor and let P be an ideal of A. Then P is called a *prime ideal* if, whenever I and J are ideals of A such that $I \cdot J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$ (see also definition (8.14)).

(7.1.7) EXAMPLE. Let A be a Green functor and let M be a simple left-A-module. Then $\mathrm{Ann}_A(M)$ is prime. Indeed, let I and J be two ideals of A such that $I \not\subseteq \mathrm{Ann}_A(M)$ and $J \not\subseteq \mathrm{Ann}_A(M)$. From the simplicity of M it follows that $I \cdot M = M$ and $J \cdot M = M$. It follows that $(I \cdot J) \cdot M = I \cdot (J \cdot M) = I \cdot M = M$; hence $I \cdot J \not\subseteq \mathrm{Ann}_A(M)$. It follows easily, from the results from chapter 9, that the ring $\left(A/\mathrm{Ann}_A(M)\right)(G)$ has characteristic zero or a prime number p. We refer to the characteristic of the ring $\left(A/\mathrm{Ann}_A(M)\right)(G)$ as the *integral characteristic* of M.

(7.1.8) THEOREM (Induction Theorem for Simple Left-A-Modules).

Let A be a Green functor and let M be a simple left-A-module which is H-characteristic.

(1) If the integral characteristic of M is $p > 0$, then M is G/H^p-projective.

(2) If A is commutative and the integral characteristic of M is zero, then M is G/H-projective.

PROOF. *(1)* Follows immediately from theorem (6.2.4) *(2)*.

(2) Since A is commutative, it follows, by proposition (7.1.5) *(3)*, that

$$\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{eq}(\overline{m_H}))}$$

for some maximal ideal $\overline{m_H}$ of $\overline{A(H)}$. Since $\overline{A(H)}/\overline{m_H}$ is a field of characteristic zero, it follows, by proposition (9.1.12), that the Green functor $A/\mathrm{Ann}_A(M)$ is G/H-projective. Since M is an $A/\mathrm{Ann}_A(M)$-module, it follows, by theorem (2.6), that M is G/H-projective as well. $\triangle$

7.2. CHARACTERIZATION THEOREMS FOR $Jac(A)$.

(7.2.1) DEFINITION. Let A be a Green functor. The *left Jacobson radical* $Jac(A)$ of A is the ideal

$$Jac(A) = \bigcap_{M} \mathrm{Ann}_A(M), \tag{7.8}$$

where the intersection is taken over all the simple left-A-modules M.

If $H \in \mathcal{P}(A)$, let $\overline{Jac_H}$ be the Jacobson radical of the ring $\overline{A(H)}$. The following result gives an explicit formula for $Jac(A)$.

(7.2.2) PROPOSITION.

Let A be a Green functor. Then

$$Jac(A) = \bigcap_{H \in \mathcal{P}(A)} A_{(H,\, \overline{Jac_H})}. \tag{7.9}$$

PROOF. From proposition (7.1.5) *(2)*, we conclude that

$$Jac(A) \supseteq \bigcap_{H \in \mathcal{P}(A)} A_{(H,\, \overline{Jac_H})}. \tag{7.10}$$

From proposition (7.1.5) *(1)*, we conclude that

$$Jac(A) \subseteq \bigcap_{H \in \mathcal{P}(A)} \bigcap_{\overline{m} \in \mathrm{Max}(\overline{A(H)})} A_{(H,\, \mathrm{eq}(\overline{m}))}. \tag{7.11}$$

For $H \in \mathcal{P}(A)$, it is clear that

$$\overline{Jac_H} = \bigcap_{\overline{m} \in \mathrm{Max}(\overline{A(H)})} \overline{m}. \tag{7.12}$$

We use formula (7.12) and proposition (6.1.21) to conclude that, for $H \in \mathcal{P}(A)$,

$$A_{(H,\, \overline{Jac_H})} = \bigcap_{\overline{m} \in \mathrm{Max}(\overline{A(P)})} A_{(H,\, \mathrm{eq}(\overline{m}))}. \tag{7.13}$$

Using formula (7.13), the containment (7.11) becomes

$$Jac(A) \subseteq \bigcap_{H \in \mathcal{P}(A)} A_{(H,\, \overline{Jac_H})}. \tag{7.14}$$

Now the theorem follows from formuale (7.10) and (7.14). $\triangle$

Using formula (7.9), we conclude that, if we define the *right Jacbson radical* of A as the intersection of the annihilators of all the simple right-A-modules, then the left and the right Jacobson radicals coincide. From now on, we refer to $Jac(A)$ as the *Jacobson radical* of A.

Let M be a left-A-module. Recall that $[G\backslash\mathcal{P}(M)]$ is a set of representatives for the conjugacy classes of subgroups in $\mathcal{P}(M)$. Assume that, for every $H \in [G\backslash\mathcal{P}(M)]$, we are given a W_GH-invariant submodule $\overline{N_H}$ of $\overline{M(H)}$. From definition (6.1.14) we know that

$$M_{(H,\,\overline{N_H})} = \operatorname{Ker} n_H, \qquad \text{where} \qquad n_H : M \longrightarrow J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right). \tag{7.15}$$

Here the map n_H is the one given by formula (6.5). By adding all the morphisms n_H, for $H \in [G\backslash\mathcal{P}(M)]$, we obtain a morphism

$$\mu : M \longrightarrow \bigoplus_{H\,\in\,[G\backslash\mathcal{P}(M)]} J_{G/H}\left(\frac{\overline{M(H)}}{\overline{N_H}}\right), \tag{7.16}$$

and

$$\operatorname{Ker} \mu = \bigcap_{H\,\in\,[G\backslash\mathcal{P}(M)]} M_{(H,\,\overline{N_H})}. \tag{7.17}$$

We now give a second description of $Jac(A)$.

(7.2.3) PROPOSITION.

$Jac(A) = \operatorname{Ker} j$, where j is the natural map

$$j : A \longrightarrow \bigoplus_{H\,\in\,[G\backslash\mathcal{P}(A)]} J_{G/H}\left(\frac{\overline{A(H)}}{\overline{Jac_H}}\right). \tag{7.18}$$

PROOF. From formulae (7.17) and (7.9) it follows that

$$\operatorname{Ker} j = \bigcap_{H\in[G\backslash\mathcal{P}(A)]} A_{(H,\,\overline{Jac_H})} = \bigcap_{H\in\mathcal{P}(A)} A_{(H,\,\overline{Jac_H})} = Jac(A). \qquad \triangle$$

When A is a commutative Green functor, there is a similar description of the nilradical of A. Let A be a commutative Green functor. For $H \in S(G)$ let $Nil(A(H))$ be the nilradical of the ring $A(H)$. For $H \in \mathcal{P}(A)$ let $\overline{Nil_H}$ be the nilradical of the ring $\overline{A(H)}$.

(7.2.4) PROPOSITION.

If A is a commutative Green functor, then $Nil(A) = \text{Ker } n$, where n is the natural map

$$n : A \longrightarrow \bigoplus_{H \in [G \backslash \mathcal{P}(A)]} J_{G/H}\left(\frac{\overline{A(H)}}{\overline{Nil_H}}\right). \tag{7.19}$$

PROOF. For simplicity, let A_1 be the commutative Green functor appearing in the right side of the formula (7.19) (with the multiplication done component-wise). Since the rings

$$\frac{\overline{A(H)}}{\overline{Nil_H}}$$

are reduced (i.e. they have trivial nilradical), one can use formula (4.15), to conclude that the nilradical of $A_1(H)$ is trivial for all $H \in S(G)$. We conclude that $Nil(A) \subseteq \text{Ker } n$.

We now prove the reverse containement. For $H \in S(G)$, and $a \in (\text{Ker } n)(H)$, we have

$$br_K \cdot r_K^H(a) \in \overline{Nil_K}, \qquad \text{for all } K \in \mathcal{P}(A) \cap S(H). \tag{7.20}$$

It follows that, for every $K \in \mathcal{P}(A) \cap S(H)$, there exists $m_K \geq 1$ such that

$$br_K \cdot r_K^H(a^{m_K}) = 0. \tag{7.21}$$

Let $m = \max\left(m_K \mid K \in \mathcal{P}(A) \cap S(H)\right)$. Then

$$br_K \cdot r_K^H(a^m) = 0, \qquad \text{for all } K \in \mathcal{P}(A) \cap S(H).$$

Hence $a^m \in (\text{Ker } \beta_A)(H)$, where $\beta_A : A \longrightarrow TA$ is the morphism given by the formula (4.14). Since Ker β_A is a nilpotent ideal of A (thanks to proposition (4.2.1) *(2)*), we conclude that a is a nilpotent element of $A(H)$. $\triangle$

(7.2.5) PROPOSITION.

If A is a commutative Green functor, then

$$Nil(A) = \beta_A^{-1}(Nil(TA)). \tag{7.22}$$

PROOF. It is clear that $\beta_A(Nil(A)) \subset Nil(TA)$. The reverse containment follows from the fact that Ker $\beta_A \subseteq Nil(A)$ (according to proposition (4.2.1) *(2)*). $\triangle$

7.3. PROPERTIES OF $Jac(A)$.

(7.3.1) PROPOSITION.

Let A be a Green functor, and let I be a two-sided ideal in A. If I is nilpotent then $I \subseteq Jac(A)$.

PROOF. If not, there exists a simple left-A-module M such that $I \not\subseteq \mathrm{Ann}_A(M)$. From the simplicity of M we conclude that $I \cdot M = M$. Hence $M = I \cdot M = ... = I^n \cdot M$ for all $n > 0$. Since I is nilpotent, we conclude that $M = 0$, which is a contradiction.

(7.3.2) PROPOSITION.

If A is a commutative Green functor, then $Nil(A) \subseteq Jac(A)$.

PROOF. Let $H \in S(G)$, and let $a \in Nil(A(H))$. Consider the ideal $I = A\langle a \rangle$. Using theorem (5.12) and proposition (5.2) *(7)*, we conclude that I is a nilpotent ideal. From proposition (7.3.1), it follows that $I \subseteq Jac(A)$, hence $a \in Jac(A)(H)$. $\triangle$

(7.3.3) DEFINITION.

A left-A-module is called *finitely generated* if there exists a surjection of A modules

$$A_X \longrightarrow M \qquad \text{for some } X \in G - Sets. \tag{7.23}$$

Equivalently, M is finitely generated if $M = A\langle m \rangle$ for some $m \in M(X)$.

(7.3.4) LEMMA.

Let A be a Green functor. Suppose that M is a non-zero finitely generated left-A-module. If N is a proper submodule of M, then M has a proper maximal submodule containing N.

PROOF. Denote by $\mathcal{S}$ the set of all proper submodules of M containing N. We use the fact that M is finitely generated to conclude that $\mathcal{S}$ is inductively ordered. Since $S \neq \emptyset$, (because $N \in S$), the result follows from Zorn's lemma. $\triangle$

(7.3.5) NAKAYAMA LEMMA.

Let A be a Green functor. Suppose that M is a finitely generated left-A-module. If $Jac(A) \cdot M = M$, then $M = 0$.

PROOF. Assume $M \neq 0$. From lemma (7.3.4), it follows that there exists $N \subset M$, such that N is maximal. Since M/N is simple, we conclude that $Jac(A) \subseteq \mathrm{Ann}_A(M/N)$, hence $M = Jac(A) \cdot M \subseteq N$. $\triangle$

Let A be a Green functor and let $H \in S(G)$. We investigate the relation between $Jac(A)(H)$ and $Jac(A(H))$. We need some preliminary results about the units of the rings $A(H)$. Let $U(A)(H)$ be the group of units of $A(H)$. For $H \in \mathcal{P}(A)$, let $\overline{U_H}$ be the group of units of the ring $\overline{A(H)}$. The following theorem describes the relation between $U(A)(H)$ and $U(TA)(H)$.

(7.3.6) PROPOSITION.

Let A be a Green functor. Then

$$U(A)(H) = \beta^{-1}(U(TA)(H)). \tag{7.24}$$

PROOF. Notice that $\beta(U(A)(H)) \subseteq U(TA)(H)$. We prove the reverse containment. From proposition (4.2.1) *(1)*, we know that $\mathcal{P}(A) = \mathcal{P}(TA)$. Hence, if H does not contain a primordial subgroup then $A(H) = (TA)(H) = 0$. If $H \in \text{Min } \mathcal{P}(A)$, then the morphism $\beta(H)$ is the identity of $A(H)$. Inductively, we assume that $H \in S(G)$, and

$$U(A)(K) \supseteq \beta^{-1}(U(TA)(H)), \qquad \text{whenever } K < H. \tag{7.25}$$

Let $u \in A(H)$ such that $\beta(H)(u)$ is a unit in $(TA)(H)$. We show that u has a right inverse in $A(H)$. First, notice that there exists a relation of the form

$$1_{A(H)} - uv = \sum_{K<H} t_K^H \, a_K, \qquad \text{for some } v \in A(H) \text{ and } \ a_K \in A(K). \tag{7.26}$$

Indeed if $H \notin \mathcal{P}(A)$, then $1_{A(H)} \in Tr_A(H)$. Hence there exists a relation (7.26) with $v = 0$. If $H \in \mathcal{P}(A)$, we use the fact that

$$br_H(u) \in \overline{U_H},$$

to conclude that $1 - uv \in Tr_A(H)$, for some $v \in A(H)$. This leads again to a relation of the form (7.26).

We use formula (7.25) to conclude that, for each $K < H$, there exists a $v_K \in A(K)$ such that $r_K^H(u) \cdot v_K = 1_{A(K)}$. Write $a_K = r_K^H(u) \cdot (v_K a_K)$. Using the Frobenius axiom, equation (7.26) becomes

$$1_{A(H)} - uv = \sum_{K<H} t_K^H\Big(r_H^K(u) \cdot (v_K a_K)\Big) = u \cdot w,$$

where $w = \sum_{K<H} t_K^H(v_K a_K)$. Hence $1_{A(H)} = u(v + w)$. The fact that u has a left inverse follows similarly. $\triangle$

As a corollary to (7.3.6), we prove the following relation between $Jac(A)$ and the Jacobson radicals $Jac(A(H))$ of $A(H)$, for $H \in S(G)$.

(7.3.7) THEOREM.

Let A be a Green functor. Then, for all $H \in S(G)$,

$$\big(Jac(A)\big)(H) \subseteq Jac(A(H)).$$

Moreover, if I is an ideal of A such that $I(H) \subseteq Jac(A(H))$, for all $H \in \mathcal{P}(A)$, then $I \subseteq Jac(A)$. Thus, $Jac(A)$ is the largest ideal I with the property that $I(H) \subseteq Jac(A(H))$ for all $H \in S(G)$.

PROOF. We show that $\big(Jac(A)\big)(H) \subseteq Jac(A(H))$ for all $H \in S(G)$. Assume that $\mathcal{P}(A) \cap S(H) \neq \emptyset$ (otherwise $A(H)$ is trivial). Let $a \in \big(Jac(A)\big)(H)$, and let $x \in A(H)$. We use proposition (7.2.3) to conclude that

$$br_K \cdot r^H_K(ax) \in \overline{Jac_K}, \qquad \text{for all } K \in \mathcal{P}(A) \cap S(H). \tag{7.27}$$

We conclude that

$$1_{\overline{A(K)}} - br_K \cdot r^H_K(ax) = br_K \cdot r^H_K(1_{A(H)} - ax) \in \overline{U_K}, \qquad \text{for all } K \in \mathcal{P}(A) \cap S(H). \tag{7.28}$$

But the above containment shows that $\beta_H(1_{A(H)} - ax) \in U(TA)(H)$. From proposition (7.3.6), we conclude that $1_{A(H)} - ax \in U(A)(H)$. Since x was arbitrary in $A(H)$, we conclude that $a \in Jac(A(H))$.

We now prove that, if I is an ideal such that $I(H) \subseteq Jac(A(H))$, for all $H \in \mathcal{P}(A)$, then $I \subseteq Jac(A)$. For $H \in \mathcal{P}(A)$, let $J(H)$ be the ideal of $A(H)$ containing $Tr_A(H)$, such that

$$\frac{J(H)}{Tr_A(H)} = \overline{Jac_H}, \tag{7.29}$$

Notice that $Jac(A(H)) \subseteq J(H)$ for all $H \in \mathcal{P}(A)$ (because $J(H)$ is the intersection of some of the maximal ideals of $A(H)$, namely the ones containing $Tr_A(H)$, whereas $Jac(A(H))$ is the intersection of all the maximal ideals of $A(H)$). Since I is functorial, we conclude that

$$I(H) \subseteq \bigcap_{K \in \mathcal{P}(A) \cap S(H)} (r^H_K)^{-1}(Jac(A(K))) \subseteq \bigcap_{K \in \mathcal{P}(A) \cap S(H)} (r^H_K)^{-1}(J(H)). \tag{7.30}$$

Using formula (6.8), we conclude that, for $K \in \mathcal{P}(A) \cap S(H)$,

$$\bigcap_{{}^gK \subseteq H} (r^H_{{}^gK})^{-1}(J({}^gK)) = A_{(K,\, \overline{Jac_K})}(H). \tag{7.31}$$

Since $A_{(K,\,\overline{Jac_K})}(H) = A(H)$, when K is not subcojugate to H (because $A_{(K,\,\overline{Jac_K})}$ is K-cocharacteristic), from the containment (7.30), we conclude that

$$I(H) \subseteq \bigcap_{K\in\mathcal{P}(A)} A_{(K,\,\overline{Jac_K})}(H) = \big(Jac(A)\big)(H), \qquad \text{for all } H \in \mathcal{P}(A). \tag{7.32}$$

The last equality above is formula (7.9). The containment $I \subseteq Jac(A)$ follows now from corollary (5.10) *(i)*. $\triangle$

(7.3.8) COROLLARY.

Let X be a finite G-set. Then

(1) An element a of $A(X)$ is in $Jac(A)(X)$ if and only if $1_{A(Y\times X)} - b\times a \in U(A)(Y\times X)$, for all $Y \in G$-Set and all $b \in A(Y)$.

(2) An element a of $A(X)$ is in $Jac(A)(X)$ if and only if $1_{A(X\times Z)} - a\times c \in U(A)(X\times Z)$, for all $Z \in G$-Set and all $c \in A(Z)$.

(3) An element a of $A(X)$ is in $Jac(A)(X)$ if and only if $1_{A(Y\times X\times Z)} - b\times a\times c \in U(A)(Y\times X\times Z)$, for all $Y,\ Z \in G$-Set and all $b \in A(Y)$ and $c \in A(Z)$.

PROOF. *(1)* For $X \in G$-*Set*, let $J'(X)$ be the subset of $A(X)$ consisting of all elements a such that $1_{A(Y\times X)} - b\times a \in U(A)(Y\times X)$, for all $Y \in G$-Set and all $b \in A(Y)$. We show that $\big(Jac(A)\big)(X) \subset J'(X)$. Indeed, if not, let Y be a finite G-set, and let $a \in \big(Jac(A)\big)(X)$ and $b \in A(Y)$, such that $c = 1_{A(Y\times X)} - b\times a$ is not a unit in $A(Y\times X)$. Let us regard c as an element in $A_{Y\times X}(G)$. From corollary (5.8) *(2)*, we conclude that $A\langle c\rangle$ is a proper left submodule of $A_{Y\times X}$. Since $A_{Y\times X}$ is a finitely generated A module, we use lemma (7.3.4) to conclude that there exists a maximal left submodule N of $A_{Y\times X}$ containing c. Let M be the simple left-A-module $A_{Y\times X}/N$. Since $c = 1_{A(Y\times X)} - b\times a \in \mathrm{Ann}_A(M)$, and $a \in \big(Jac(A)\big)(X) \in \big(\mathrm{Ann}_A(M)\big)(X)$, it follows that $M = 0$, contradicting the simplicity of M.

Now let $H \in \mathcal{P}(A)$. We show that $J'(H) \subseteq \big(Jac(A)\big)(H)$. Let $a \in J'(H)$, $K \in \mathcal{P}(A) \cap S(H)$, and $b \in A(K)$. Since

$$1_{A(G/K\times G/H)} - b\times a \in U(A)(G/K\times G/H),$$

we use formula (5.5) to conclude that $(1_{A(K)} - b\cdot r^H_K(a)) \in U(A)(K)$. Since b was arbitrary in $A(K)$, we conclude that $r^H_K(a) \in Jac(A(K))$. It follows that

$$J'(K) \subseteq \bigcap_{K\in\mathcal{P}(A)\cap S(H)} (r^H_K)^{-1}\,\big(Jac(A(K))\big).$$

From the formulae (7.30). (7.31) and (7.32), we know that this last ideal is contained in $(Jac(A))(H)$.

Now assume $H \notin \mathcal{P}(A)$. It is easy to see that the family $(J'(K))_{K\in S(G)}$, is closed under restrictions. Hence

$$J'(H) \subseteq \bigcap_{K\in\mathcal{P}(A)\cap S(H)} (r^H_K)^{-1}(J'(K)) \subseteq \bigcap_{K\in\mathcal{P}(A)\cap S(H)} (r^H_K)^{-1}(Jac(A)(K)) = (Jac(A))(H),$$

where the last equality follows from proposition (4.1.6) *(3)*.

(2) Follows from a similar argument as *(1)* using the fact that the left Jacobson radical of A coincides with the right Jacobson radical of A.

(3) Immediate consequence of *(1)* and *(2)*. $\triangle$

(7.3.9) COROLLARY.

Let X be a finite G-Set. Then the submodules $Jac(A_X)$ and $Jac(A)_X$ of A_X are equal.

PROOF. We first show that $Jac(A)_X \subset Jac(A_X)$. Let Y be a finite G-set, and let $a \in (Jac(A)_X)(Y) = (Jac(A))(X\times Y) \subseteq A(X\times Y)$. From corollary (7.3.8) *(2)*, we conclude that, for all G-sets Z, and all $b \in A(Z)$, $1_{A(X\times Y\times Z)} - a\times b$ is a unit in $A(X\times Y\times Z)$. In particular, this holds for all G-sets of the form $X\times Z$, and for all $b\in A(X\times Z) = A_X(Z)$. From corollary (7.3.8) *(2)*, we conclude that $a \in (Jac(A_X))(Y)$.

Conversely, let $a \in (Jac(A_X))(Y)$. Let $Z \in$ *G-Set*, and let $b \in A(Z)$. Let us consider the element $c = 1_{A(X)} \times b \in A(X\times Z) = A_X(Z)$. Notice that

$$1_{A(X\times Y\times X\times Z)} - a\times c \in U(A)(X\times Y\times X\times Z).$$

From theorem (2.4), it follows easily that $A(X\times Y\times Z)$ can be regarded as a direct summand in $A(X\times Y\times X\times Z)$. We conclude that

$$1_{A(X\times Y\times Z)} - a\times b \in U(A)(X\times Y\times Z).$$

Therefore, from corollary (7.3.8) *(2)*, we conclude that $a \in Jac(A)_X(Y)$. $\triangle$

We compute the Jacobson radicals of some classical Green functors. We need the following lemma.

(7.3.10) LEMMA.

Suppose that A is a commutative Green functor

(1) If $\overline{Jac_H} = \overline{Nil_H}$, for all $H \in \mathcal{P}(A)$, then $Jac(A) = Nil(A)$.

(2) If $\overline{Jac_H} = 0$ (or $\overline{Nil_H} = 0$) for all $H \in \mathcal{P}(A)$, then $Jac(A) = \text{Ker } \beta_A$ (respectively $Nil(A) = \text{Ker } \beta_A$), where β_A is the canonical morphism from A to TA. The statement about $Jac(A)$ remains true if A is an arbitrary Green functor (not necessarily commutative).

PROOF. *(1)* Follows immediately from formulae (7.18) and (7.19).

(2) Follows from formulae (7.18) (respectively (7.19)), (4.22) and (4.23). △

(7.3.11) EXAMPLE. Let R be a ring such that $Jac(R) = 0$. Let B be the Burnside ring functor for G over R. Since $\overline{B(H)} = R$, for all $H \in S(G)$, we use lemma (7.3.10) *(2)* to conclude that $Jac(B) = \text{Ker } \beta_B$. By proposition (4.2.1) *(2)*, we know that Ker β_B is nilpotent. Hence $Jac(B) = Nil(B)$. It follows that, if $R = \mathbf{Z}$, then $Jac(B) = 0$, because the rings $B(H)$ have trivial nilradical. If R is a field of characteristic $p \not| |G|$, then, from corollary (4.2.2) *(1)*, we know that Ker $\beta_B = 0$. Hence $Jac(B) = 0$ in this case. Finally, if R is a field of characteristic $p \mid |G|$, then B has nilpotent elements, hence $Jac(B) \neq 0$.

(7.3.12) EXAMPLE. Let $R_{\mathbf{C}}$ be the character ring Green functor. We check that $\overline{Jac_H} = \overline{Nil_H}$, for all $H \in \mathcal{P}(R_{\mathbf{C}})$.

If H is cyclic, then, $\overline{R_{\mathbf{C}}} = \mathbf{Z}[\zeta_H]$, where ζ_H is a primitive root of order $|H|$ of unity (see [T4], exercise 54.3, p.513). Since $\mathbf{Z}[\zeta_H]$ is a ring of algebraic integers, it follows that it has Krull dimension 1. Since $\mathbf{Z}[\zeta_H]$ is a domain, we conclude that every non-zero prime ideal in $\mathbf{Z}[\zeta_H]$ is maximal. Hence $Jac(\mathbf{Z}[\zeta_H]) = Nil(\mathbf{Z}[\zeta_H]) = 0$.

If $H \in \mathcal{P}(A)$ is non-cyclic, then we use the Artin Induction Theorem (see example (2.10) *(2)*), to conclude that $\overline{R_{\mathbf{C}}(H)}$ is a ring in which every element has finite torsion. Since $\overline{R_{\mathbf{C}}(H)}$ is a finitely generated $\mathbf{Z}$-module, we conclude that $\overline{R_{\mathbf{C}}(H)}$ is a finite ring. In particular, $\overline{R_{\mathbf{C}}(H)}$ is artinian, hence $\overline{Jac_H} = \overline{Nil_H}$.

From lemma (7.3.10) *(1)*, it follows that $Jac(R_{\mathbf{C}}) = Nil(R_{\mathbf{C}})$. Since the rings $R_{\mathbf{C}}(H)$ have trivial nilradicals for $H \in S(G)$, we conclude that $Jac(R_{\mathbf{C}}) = 0$.

Similar considerations apply to $R_{\mathbf{C}} \otimes \mathbf{Q}$.

We give an example which shows that the Jacobson radical of a Green functor A need not be the intersection of all the maximal ideals of A.

(7.3.13) EXAMPLE. Let p be a prime number, and let G be a non-trivial p-group. Let S be the trivial $\mathbf{Z}[G]$-algebra $\mathbf{Z}_p$. Let $A = FP_S$. One can check easily that $\mathcal{P}(A) = S(G)$, and that every proper ideal of A is contained in $A_{(G,0)}$. Hence, the only maximal ideal of A is $A_{(G,0)}$. On the other hand, let M be the A-module constructed at example (6.2.3). It is clear that M is simple. From example (6.2.3), we know that $\mathrm{Ann}_A(M) = 0$. Hence $0 = Jac(A) \neq A_{(G,0)}$.

7.4. TOTALLY DECOMPOSABLE GREEN FUNCTORS.

(7.4.1) DEFINITION. A Mackey functor M such that the map β_M given by formula (4.14) (or (4.23)) is an isomorphism is called *totally decomposable* (see also definition (4.2.3)).

(7.4.2) EXAMPLE. Assume that A is a Green functor for G over R such that $|G|$ is invertible in R (or $A(G)$). From corollary (4.2.1), it follows that A is totally decomposable. Hence the results of this section apply to these Green functors.

It this section we show that if A is a totally decomposable commutative Green functor then $Jac(A)(H) = Jac(A(H))$, for all $H \in S(G)$. Moreover, we show that in this case $Jac(A)$ is the intersection of all maximal ideals of A. We begin with some preliminary results.

Let A be a Green functor and let M be a left-A-module. For each $H \in \mathcal{P}(M)$, $\mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})$ is a proper W_GH-invariant ideal of $\overline{A(H)}$. Let

$$\alpha_H : A \longrightarrow J_{G/H}\left(\frac{\overline{A(H)}}{\mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})}\right) \tag{7.33}$$

be the canonical map given by formula (6.5). Let

$$\alpha = \bigoplus_{H\in[G\backslash\mathcal{P}(M)]} \alpha_H : A \longrightarrow \bigoplus_{H\in[G\backslash\mathcal{P}(M)]} J_{G/H}\left(\frac{\overline{A(H)}}{\mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})}\right) \tag{7.34}$$

be the direct sum of the maps α_H. In the next result, we bound the ideal $\mathrm{Ann}_A(M)$ in terms of the ideals $\mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})$ of $\overline{A(H)}$.

(7.4.3) PROPOSITION.

Let A be a Green functor and let M be a left-A-module. Let α be the map given by formula (7.34). *Denote pl_M by m (for the definition of pl_M, see* (6.1.27)*). Then*

$$(\mathrm{Ker}\ \alpha)^m \subseteq \mathrm{Ann}_A(M) \subseteq \mathrm{Ker}\ \alpha. \tag{7.35}$$

PROOF. It is clear that $\mathrm{Ann}_A(M) \subseteq \mathrm{Ker}\ \alpha$. We prove the other containment. Let $H \in \mathcal{P}_i(M)$, for some $i \geq 1$ (see definition (6.1.27)). We use induction over i, the height of H, to show that

$$\Big((\mathrm{Ker}\ \alpha)(H)\Big)^i \subseteq \mathrm{Ann}_{A(H)}(M(H)). \tag{7.36}$$

It is clear that formula (7.36) is, in fact, an equality when $i = 1$. Assume $i > 1$. Let $H \in \mathcal{P}_i(M)$, and let $a \in (\mathrm{Ker}\ \alpha)(H)$. Since

$$br_H(a) \in \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)}),$$

it follows that

$$a \cdot m \in Tr_M(H), \qquad \text{for all } m \in M(H). \tag{7.37}$$

Fix $m \in M(H)$, and use proposition (4.1.6) *(1)* to conclude that

$$a \cdot m = \sum_{\substack{K<H \\ K \in \mathcal{P}(M)}} t_K^H(m_K), \qquad \text{for some } m_K \in M(K). \tag{7.38}$$

Let a_1, ..., a_i, be i elements in $(\mathrm{Ker}\ \alpha)(H)$ such that $a_i = a$. We use the induction assumption and the Frobenius axiom to conclude that

$$(\prod_{j=1}^{i} a_j) \cdot m = \sum_{\substack{K<H \\ K \in \mathcal{P}(M)}} t_K^H\Big((\prod_{j=1}^{i-1} r_K^H(a_j)) \cdot m_K\Big) = 0. \tag{7.39}$$

Hence containment (7.36) holds. From (7.36), it follows that

$$\Big((\mathrm{Ker}\ \alpha)(H)\Big)^m \subseteq \mathrm{Ann}_{A(H)}(M(H)), \qquad \text{for all } H \in \mathcal{P}(M), \text{ and } m = pl_M. \tag{7.40}$$

One can use proposition (4.1.6) *(1)*, the fact that Ker α is functorial, and the Frobenius axiom, to conclude that containment (7.40) holds for all $H \in \mathcal{S}(G)$, i.e.

$$\Big((\mathrm{Ker}\ \alpha)(H)\Big)^m \subseteq \mathrm{Ann}_{A(H)}(M(H)), \qquad \text{for all } H \in \mathcal{S}(G). \tag{7.41}$$

One can use the Frobenius axiom to conclude that the family $\big(\mathrm{Ann}_{A(H)}(M(H))\big)_{H \in \mathcal{S}(G)}$ is closed under transfers. Finally, one can use formula (3.16) to conclude that

$$(\mathrm{Ker}\ \alpha)^m(H) = \sum_{K \leq H} t_K^H\bigg(\Big((\mathrm{Ker}\ \alpha)(K)\Big)^m\bigg) \subseteq \mathrm{Ann}_{A(H)}(M(H)), \quad \text{for all } H \in \mathcal{S}(G). \tag{7.42}$$

One can easily check that $\mathrm{Ann}_A(M)$ is the largest ideal J such that

$$J(H) \subseteq \mathrm{Ann}_{A(H)}(M(H)) \qquad \text{for all } H \in S(G).$$

Now the containment

$$(\mathrm{Ker}\ \alpha)^m \subseteq \mathrm{Ann}_A(M),$$

follows from formula (7.42). $\triangle$

Proposition (7.4.3) has the following immediate corollary.

(7.4.4) COROLLARY.

Suppose that A is a commutative Green functor. If M is an A-module such that $\mathrm{Ann}_A(M)$ is radical then $\mathrm{Ann}_A(M) = \mathrm{Ker}\ \alpha$.

From proposition (7.4.3), it also follows that one can derive an induction theorem for M from the structure of the left-$\overline{A(H)}$-modules $\overline{M(H)}$, for $H \in \mathcal{P}(M)$.

(7.4.5) PROPOSITION.

Let A be a Green functor, M be a left-A-module and X be a G-set. Assume that the Green functors

$$\frac{A}{A_{\left(H,\ \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})\right)}}$$

are X-projective for all $H \in \mathcal{P}(M)$. Then M is X-projective.

PROOF. Let α be the morphism given by formula (7.34). Since

$$\mathrm{Ker}\ \alpha = \bigcap_{H \in \mathcal{P}(M)} A_{\left(H,\ \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)})\right)}$$

we use proposition (4.1.15) *(1)*, to conclude that $A/(\mathrm{Ker}\ \alpha)$ is X-projective. From proposition (4.1.15)*(2)*, we conclude that, for $m = pl_M$, $A/(\mathrm{Ker}\ \alpha)^m$ is also X-projective. From proposition (7.4.3), we conclude that $A/\mathrm{Ann}_A(M)$ is a left-$A/(\mathrm{Ker}\ \alpha)^m$-module. From theorem (2.6), we conclude that $A/\mathrm{Ann}_A(M)$ is X-projective. Finally, since M is a left-$A/\mathrm{Ann}_A(M)$-module, we use theorem (2.6) to conclude that M is X-projective. $\triangle$

We also need the following two lemmas.

(7.4.6) LEMMA.

(1) Let $H \in S(G)$ and let W_1 be a subgroup of W_GH. Let N_1 be the subgroup of N_GH such that $W_1 = N_1/H$. Let $M \in Mack_R(W_1)$. Then

$$\mathcal{P}\Big((\mathrm{Inf}_{W_1}^{N_1} M)\uparrow_{N_1}^{G}\Big) = \mathrm{Cl}_G\Big(\{K \in S(N_1) \mid H \triangleleft K \text{ and } K/H \in \mathcal{P}(M)\}\Big). \tag{7.43}$$

(2) If $\overline{M}$ is an $R[W_GH]$-module, then

$$\mathcal{P}(J_{G/H}(\overline{M})) = \mathrm{Cl}_G\Big(\{K \in S(G) \mid H \triangleleft K \text{ and } K/H \in \mathcal{P}(FP_{\overline{M}})\}\Big). \tag{7.44}$$

PROOF. Formula (7.43) follows immediately from definition (4.2.4) and proposition (4.1.9) *(1)*. Formula (7.44) is an immediate consequence of (7.43). △

(7.4.7) LEMMA.

Let $H \in S(G)$, and assume that $\overline{A}$ is an $R[W_GH]$-algebra. The following assertions are equivalent:

(1) The functor $J_{G/H}(\overline{A}) \in Green_R(G)$ is G/H-projective.

(2) The trivial subgroup 1 is the only primordial subgroup of the functor $FP_{\overline{A}}$ for W_GH.

(3) The $R[W_GH]$-algebra $\overline{A}$ is projective; i.e. there exists $a \in \overline{A}$, such that

$$1_{\overline{A}} = \sum_{g \in W_GH} g \cdot a. \tag{7.45}$$

PROOF. Notice that the functor $J_{G/H}(\overline{A})$ is G/H-projective, if and only if

$$\mathcal{P}(J_{G/H}(\overline{A})) = [H].$$

Now (1) ⇔ (2) follows from lemma (7.4.6)*(2)*.

(2) ⇔ (3) follows from theorem (2.6) and the fact that the transfer map $t_1^{W_GH}$ of the functor $FP_{\overline{A}}$ is exactly the trace map from $\overline{A}$ to $\overline{A}^{W_GH}$ (see example (1.1.2)). △

(7.4.8) PROPOSITION (Totally Decomposable Commutative Green Functors).

Let A be a commutative Green functor. The following assertions are equivalent:

(1) If $H \in \mathcal{P}(A)$, then $J_{G/H}(\overline{A(H)})$ is G/H-projective.

(2) If $H \in \mathcal{P}(A)$, then every H-characteristic A-module is G/H-projective.

(3) If M is an A-module, then M is projective relative to $\mathcal{P}(M)$.

(4) Every simple A-module M is G/H-projective, where $[H] = \mathcal{P}(M)$.

(5) For $H \in \mathcal{P}(A)$, the $R[W_G H]$-algebra $\overline{A(H)}$ is projective.

(6) If M is an A-module, then $\beta_M : M \longrightarrow T_M$ is bijective.

(7) If M is an A-module, then $\beta_M : M \longrightarrow T_M$ is surjective.

(8) The map $\beta_A : A \longrightarrow TA$ is surjective.

(9) The map $\beta_A : A \longrightarrow TA$ is bijective.

(10) The map $\beta_{A/Jac(A)} : A/Jac(A) \longrightarrow T(A/Jac(A))$ is surjective.

(11) The map $\beta_{A/Jac(A)} : A/Jac(A) \longrightarrow T(A/Jac(A))$ is surjective.

PROOF. First we show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Let $H \in \mathcal{P}(A)$, and suppose that M is an H-characteristic A module. Since $M(H)$ is an $W_G H$-equivariant $\overline{A(H)}$-module, it follows, from proposition (6.1.6), that the pairing

$$\overline{A(H)} \otimes M(H) \longrightarrow M(H)$$

corresponds to a natural pairing

$$J_{G/H}(\overline{A(H)}) \,\square\, J_{G/H}(M(H)) \longrightarrow J_{G/H}(M(H)).$$

Since $J_{G/H}(M(H))$ is an $J_{G/H}(\overline{A(H)})$-module, from theorem (2.6), we conclude that $J_{G/H}(M(H))$ is G/H-projective. In particular $\mathcal{P}\big(J_{G/H}(M(H))\big) = [H]$. We end the proof by showing that the injective map

$$j_{G/H} : M \longrightarrow J_{G/H}(M(H))$$

is, in fact, an isomorphism. Indeed, notice that this map is surjective at H. Since the only primordial subgroups of $J_{G/H}(M(H))$ are conjugate to H, it follows, from corollary (4.1.5)*(2)*, that $j_{G/H}$ is surjective.

(2) $\Rightarrow$ (3). Let M be an A-module, and let $H \in \mathcal{P}(M)$. Since

$$\frac{A}{A_{(H,\ \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)}))}}$$

is H-characteristic, we conclude from *(2)*, that it is G/H-projective. In particular, the above Green functor is also $X = \coprod_{H \in \mathcal{P}(M)} G/H$-projective. From proposition (7.4.5), we conclude that M is projective relative to $\mathcal{P}(M)$..

(3) $\Rightarrow$ (4). From theorem (7.1.3), we know that, if M is simple, then $\mathcal{P}(M) = [H]$ for some $H \in \mathcal{P}(A)$. Now (4) follows immediately from (3).

(4) $\Rightarrow$ (5). Let $H \in \mathcal{P}(A)$, and let $\overline{m} \in \operatorname{Max}\left(\overline{A(H)}\right)$. Notice that $\overline{A(H)}/\operatorname{eq}(\overline{m})$ is an W_GH-equivariant simple $\overline{A(H)}$-module (see also theorem (A1.17)). Let

$$M' = J_{G/H}\Big(\frac{\overline{A(H)}}{\operatorname{eq}(\overline{m})}\Big)$$

and let

$$M = A\langle M'(H)\rangle = \sum_{x \in M'(H)} A\langle x\rangle.$$

From the argument used in the proof of proposition (7.1.5) *(1)*, we know that M is simple. Since M is G/H-projective, we conclude that M is H-determined. In particular, $M = M'$. Notice that M' is, in fact, a Green functor. From lemma (7.4.7), it follows that the $R[W_GH]$-algebra $\overline{A(H)}/\operatorname{eq}(\overline{m})$ is projective. In particular

$$1_{\overline{A(H)}} - \sum_{g \in W_GH} g \cdot a \in \operatorname{eq}(\overline{m}), \qquad \text{for some } a \in \overline{A(H)}. \tag{7.46}$$

Let

$$Tr^{W_GH} = \Big\{ \sum_{g \in W_GH} g \cdot b \mid \text{for } b \in \overline{A(H)} \Big\}. \tag{7.47}$$

Notice that Tr^{W_GH} is an ideal of $\overline{A(H)}^{W_GH}$. From equation (7.46) we conclude that

$$\overline{A(H)}^{W_GH} = Tr^{W_GH} + \overline{m} \cap \overline{A(H)}^{W_GH}, \qquad \text{for all } \overline{m} \in \operatorname{Max}\left(\overline{A(H)}\right). \tag{7.48}$$

The ring $\overline{A(H)}$ is integral over $\overline{A(H)}^{W_GH}$. From theorems (A2.6) and (A2.7), it follows that for every $\overline{n} \in \operatorname{Max}\left(\overline{A(H)}^{W_GH}\right)$, there exists $\overline{m} \in \operatorname{Max}\left(\overline{A(H)}\right)$ such that $\overline{n} = \overline{m} \cap \overline{A(H)}^{W_GH}$. From equation (7.48), we conclude that

$$\overline{A(H)}^{W_GH} = Tr^{W_GH} + \overline{n}, \qquad \text{for all } \overline{n} \in \operatorname{Max}\left(\overline{A(H)}^{W_GH}\right). \tag{7.49}$$

From (7.49) it follows that $\overline{A(H)}^{W_GH} = Tr^{W_GH}$. In particular, the $R[W_GH]$-algebra $\overline{A(H)}$ is projective.

(5) $\Rightarrow$ (1). Follows from lemma (7.4.7).

So far we know that (1)-(5) are equivalent. Now we show that (4) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1).

(4) ⇒ (6). Induction over $[G\backslash\mathcal{P}(M)]$. Suppose that $[G\backslash\mathcal{P}(M)] = 1$, and let $H \in \mathcal{P}(M)$. Since M is G/H-projective, it follows that M is G/H-injective as well. Since M is obviously H-bounded, we conclude that M is H-characteristic. Since M is also G/H-projective, we conclude that M is H-determined. Hence

$$M \cong J_{G/H}(M(H)) = TM.$$

Assume that $[G\backslash\mathcal{P}(M)] > 1$, and let $K \in \operatorname{Min} \mathcal{P}(M)$. Let

$$N = \sum_{m \in M(K)} A\langle m\rangle.$$

It is clear that $[K] = \mathcal{P}(N)$. From the induction hypothesis it follows that

$$N \cong TN = J_{G/K}(M(K)).$$

Moreover, notice that $\mathcal{P}(M/N) = \mathcal{P}(M) - [K]$, hence $[G\backslash\mathcal{P}(M/N)] = [G\backslash\mathcal{P}(M)] - 1 < \mathcal{P}(M)$. If $H \in \mathcal{P}(M) - [K]$ then

$$N(H) = \sum_{{}^gK < H} t^H_{{}^gK} M({}^gK)$$

From the above formula for $N(H)$, it follows easily that

$$\overline{(M/N)(H)} = \overline{M(H)}, \qquad \text{for all } H \in \mathcal{P}(M) - [K].$$

We conclude that the sequence

$$0 \longrightarrow TN \longrightarrow TM \longrightarrow T(M/N) \longrightarrow 0 \tag{7.50}$$

is exactly the sequence

$$0 \longrightarrow J_{G/K}(M(K)) \longrightarrow \sum_{H \in [G\backslash\mathcal{P}(M)]} J_{G/H}(\overline{M(H)}) \longrightarrow \sum_{H \in [G\backslash\mathcal{P}(M)] - K} J_{G/H}(\overline{M(H)}) \longrightarrow 0.$$

Hence the sequence (7.50) is exact (and split).

Finally, in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\beta_N"'] & M \arrow[r] \arrow[d, "\beta_M"'] & M/N \arrow[r] \arrow[d, "\beta_{M/N}"'] & 0 \\
0 \arrow[r] & TN \arrow[r] & TM \arrow[r] & T(M/N) \arrow[r] & 0
\end{tikzcd}$$

both β_N and $\beta_{M/N}$ are bijective, from the induction hypothesis. It follows that β_M is bijective as well.

(6) $\Rightarrow$ (7) $\Rightarrow$ (8). Obvious.

(8) $\Rightarrow$ (1). Let $K \in \mathcal{P}(A)$. Since $J_{G/K}(\overline{A(K)})$ is an K-characteristic A-module, we conclude that if $H \in \mathcal{P}(J_{G/K}(\overline{A(K)}))$, then $H \in \mathcal{P}(A)$, and $[K] \leq [H]$. If $[K] < [H]$, the epimorphism β_A will induce an epimorphism

$$\overline{A(H)} \longrightarrow \overline{TA(H)} = \Big(\bigoplus_{\substack{L \in [G \backslash \mathcal{P}(A)] \\ [L] < [H]}} \overline{J_{G/L}(\overline{A(L)})(H)} \Big) \bigoplus \overline{A(H)}.$$

We conclude that

$$\overline{J_{G/L}(\overline{A(L)})(H)} = 0 \qquad \text{for all } L \in \mathcal{P}(M),\ L < H.$$

In particular

$$\overline{J_{G/K}(\overline{A(K)})(H)} = 0.$$

Hence $\mathcal{P}(J_{G/K}(\overline{A(K)})) = [K]$. Since $J_{G/K}(\overline{A(K)})$ is a Green functor, it follows that $J_{G/K}(\overline{A(K)})$ is G/K-projective.

So far we know that (1)-(8) are equivalent. Notice that (9) $\Rightarrow$ (8). Moreover, since (8) $\Rightarrow$ (6), it follows that (8) $\Rightarrow$ (9) (because (9) is just (6) with $M = A$). Hence, (1)-(9) are equivalent.

Finally we prove that (10), (11) and (4) are equivalent. Indeed, this follows immediately from the fact that (8), (9) and (4) are equivalent, and that there is a one-to-one correspondence between the simple A-modules and the simple $A/Jac(A)$-modules which preserves primordial subgroups and relative projectivity. $\triangle$

We give an example which shows that a simple A-module which is H-characteristic might not be G/H-projective.

(7.4.9) EXAMPLE. Let p be a prime number and let $G = \mathbf{Z}_p$. Consider $A = FP_{\mathbf{Z}_p}$, where G acts trivially on the ring $\mathbf{Z}_p$. Let $M(G) = 0$ and $M(1) = \mathbf{Z}_p$, be the A-module with the maps r_1^G and t_1^G both zero. It is clear that M is simple and 1-characteristic. However, M is not $G/1$-projective (otherwise M will be isomorphic with $J_{G/1}(M(1)) = A$).

We now give a characterization theorem for totally decomposable commutative Green functors in terms of the Jacobson ideal.

(7.4.10) THEOREM.

Let A be a commutative Green functor. The following conditions are equivalent:

(1) A is totally decomposable.

(2) The annihilator of every simple A-module is a maximal ideal.

PROOF. (1) $\Rightarrow$ (2). Let $H \in \mathcal{P}(A)$, and let $\overline{m} \in \mathrm{Max}\,\overline{A(H)}$. Then

$$\frac{A}{A_{(H,\ \mathrm{eq}(\overline{m}))}} \tag{7.51}$$

is G/H-projective. Hence

$$\frac{A}{A_{(H,\ \mathrm{eq}(\overline{m}))}} = J_{G/H}\left(\frac{A(H)}{\mathrm{eq}(\overline{m})}\right).$$

Since the above Green functor is G/H-projective, it follows, by lemma (7.4.7), that the $R[W_GH]$-algebra $A(H)/\mathrm{eq}(\overline{m})$ is projective. It is clear that this ring is W_GH-equivariantly simple. From theorem (9.1.8), it follows that the Green functor given by formula (7.51) is simple. Hence $A_{(H,\ \mathrm{eq}(\overline{m}))}$ is a maximal ideal of A. Now (2) follows from proposition (7.1.5) *(3)*.

(2) $\Rightarrow$ (1). Let M be a simple A-module, and let $H \in \mathcal{P}(M)$. Since $A/\mathrm{Ann}_A(M)$ is a simple Green functor, it follows easily that $\mathcal{P}(A/\mathrm{Ann}_A(M)) = [H]$. In particular $A/\mathrm{Ann}_A(M)$ is G/H-projective, hence M is G/H-projective. From proposition (7.4.8) we conclude that A is totally decomposable. Δ

(7.4.11) COROLLARY.

Let A be a totally decomposable commutative Green functor. Then

$$Jac(A) = \bigcap_{m \text{ maximal ideal of } A} m.$$

Thévenaz has found examples of Green functors for which $\left(Jac(A(H))\right)_{H\in S(G)}$ is not functorial (see [T4], exercise 11.3, p. 94). From theorem (7.3.7), we conclude that, if A is a Green functor such that $(Jac(A(H)))_{H\in S(G)}$ is an ideal then it must coincide with $Jac(A)$. It is natural to ask if there are general conditions under which $\left(Jac(A(H))\right)_{H\in S(G)}$ becomes functorial. Here is one situation.

(7.4.12) COROLLARY.

Let A be a totally decomposable commutative Green functor. Then

$$(Jac(A))(H) = Jac(A(H)) \quad \text{for all } H \in S(G). \tag{7.52}$$

PROOF. Since $A = TA$, we conclude, from formula (4.15), that for all $H \in S(G)$ $A(H)$ is integral over $A(G)$, via the restriction map r_H^G. It follows that the extension

$$\bar{r}_K^H : \frac{A(H)}{\operatorname{Ker} r_K^H} \longrightarrow A(K)$$

is integral for all $K < H \in S(G)$. From the properties of integral extensions (see theorem (A2.6) and (A2.7))), we conclude that, whenever $m \in \operatorname{Max}(A(K))$, $(r_K^H)^{-1}(m) \in \operatorname{Max}(A(H))$. From formulae (7.32), (7.31), (7.30) and (4.5), we conclude that

$$(Jac(A))(H) = \bigcap_{K \in \mathcal{P}(A) \cap S(H)} (r_H^H)^{-1}(J(K))$$

$$\supseteq \bigcap_{K \in \mathcal{P}(A) \cap S(H)} (r_K^H)^{-1}(Jac(A(K))) \supseteq Jac(A(H)).$$

The reverse containment follows from theorem (7.3.7). $\triangle$

For another example of commutative Green functors for which equality (7.52) holds, see corollary (8.25).

(7.4.13) PROPOSITION.

Let A be a commutative Green functor. Then A is totally decomposable if and only if TA is totally decomposable.

PROOF. If A is totally decomposable then $A \cong TA$. Hence TA is totally decomposable as well. Conversely, assume that TA is totally decomposable and let $H \in \mathcal{P}(= A) = \mathcal{P}(TA)$ (according to proposition (4.2.1) *(1)*. Then $J_{G/H}(\overline{A(H)})$ is an H-characteristic TA-module. From proposition (7.4.8) we conclude that $J_{G/H}(\overline{A(H)})$ is G/H-projective. From proposition (7.4.8) again it follows that A is totally decomposabile. $\triangle$

(7.4.14) COROLLARY.

Let A be a commutative Green functor. If A is not totally decomposable, then $T^n(A)$ is not totally decomposable, for any $n > 1$ (here $T^n(A)$ stands for the n-th fold iterated twin functor $(T \circ T \circ \ldots \circ T)(A)$).

PROOF. Immediate consequence of proposition (7.4.13). $\triangle$

8. Chain Conditions.

We begin with the following definitions.

(8.1) DEFINITION (Chain Conditions). Let A be a Green functor and let M be a left-A-module. We introduce the following chain conditions:

(ACC) Every ascending chain of submodules of M is stationary.

(DCC) Every descending chain of submodules of M is stationary.

(MAX) Every family of submodules of M has a maximal element.

(MIN) Every family of submodules of M has a minimal element.

(8.2) DEFINITION. (1) A left-A-module M is called *left-noetherian* (*left-artinian*) if M satisfies (ACC) (respectively (DCC)) on the set of all left-submodules.

(2) A left-A-module M is called *totally left-noetherian* (*totally left-artinian*) if M_X is left-noetherian (respectively left-artinian) for all $X \in G\text{-}Set$.

A Green functor A is *left-noetherian* (*left-artinian*) if A is left-noetherian (left-artinian) as a left-module over itself. One can define similarly the notions of totally left-noetherian (respectively totally left-artinian) Green functors.

Since all modules that appear in this chapter are left-A-modules, we refer to a left-noetherian (left-artinian) module as being *noetherian* (respectively *artinian*), and to a totally left-noetherian (totally left-artinian) module, as being *totally noetherian* (respectively *totally artinian*).

In this chapter we show that the totally left-noetherian (totally left-artinian) Green functors behave in the same way as the classical left-noetherian (left-artinian) rings. The noetherian (artinian) Green functors are less well behaved than the totally noetherian (totally artinian) ones. For example, a finitely generated module over a left-noetherian Green functor need not be noetherian. When M is a left-A-module we give necessary and sufficient conditions for M to be totally noetherian (totally artinian). We also give necessary and sufficient conditions for M to be noetherian (artinian). Although the left-noetherian (left-artinian) Green functors are not very well behaved, we show that they share some of the properties that left-noetherian (left-artinian) rings have. We show that a commutative Green functor A is noetherian if and only if every prime ideal of A is finitely generated (Cohen's theorem). We show that the Jacobson radical of a left-artinian Green functor is nilpotent. We also show that left-artinian Green functors are left-noetherian, and that there are only finitely many isomorphism classes of simple left-modules over such Green functors. We prove if A is a commutative artinian Green functor whose Jacobson radical is trivial, then $A(H)$ is semisimple artinian for all $H \in S(G)$. Unfortunately such Green functors are not semisimple in general. We show that if A is commutative artinian, then $Jac(A) = \left(Jac(A(H))\right)_{H \in S(G)}$.

The following proposition can be proved in the same way as its analogue from classical algebra.

(8.3) PROPOSITION.

(1) The full subcategories of noetherian, artinian, totally noetherian, and totally artinian left-A-modules are SC's (see definition (6.1.30)).

(2) M is noetherian (artinian) if and only if M satisfies (MAX) (respectively (MIN)) on the set of all submodules.

(3) If M is noetherian, then M is finitely generated.

(4) M is noetherian if and only if every submodule of M is finitely generated.

(5) If A is totally left-noetherian, then every finitely generated left-A-module is totally noetherian.

(6) Let $(N_i)_{1 \le i \le n}$, be n submodules of M such that M/N_i are noetherian (artinian), for $i = 1, 2, \ldots, n$. Then

$$\frac{M}{\bigcap_{i=1}^{n} N_i}$$

is also noetherian (artinian). A similar result holds if the assumption noetherian (artinian) is replaced with the assumption totally noetherian (respectively totally artinian).

(8.4) PROPOSITION.

If M is noetherian (artinian), then

(1) $M(G)$ is a left-noetherian (respectively left-artinian) $A(G)$-module.

(2) $\overline{M(K)}$ is a left-noetherian (respectively left-artinian) $\overline{A(K)}$-module for all $K \in \mathcal{P}(M)$.

PROOF. We will treat only the M noetherian case.

(1) Let

$$M_{1,G} \subseteq M_{2,G} \subseteq \ldots \subseteq M_{n,G} \subseteq \ldots \tag{8.1}$$

be an ascending chain of submodules of $M(G)$. Let M_i be the following submodule of M:

$$M_i = A\langle M_{i,G}\rangle = \sum_{m \in M_{i,G}} A\langle m\rangle.$$

The sequence

$$M_1 \subseteq M_2 \subseteq \ldots M_n \subseteq \ldots \tag{8.2}$$

is stationary. Since $M_i(G) = M_{i,G}$ (thanks to corollary (5.8) *(2)*), we conclude that the sequence (8.1) is stationary as well.

(2) Let $K \in \mathcal{P}(M)$. Let

$$\overline{N}_{1,K} \subseteq \overline{N}_{2,K} \subseteq \ldots \subseteq \overline{N}_{n,K} \subseteq \ldots \tag{8.3}$$

be an ascending sequence of W_GK-invariant left-$\overline{A(K)}$-submodules of $\overline{M(K)}$. Since the sequence

$$M_{(K,\,\overline{N}_{1,K})} \subseteq M_{(K,\,\overline{N}_{2,K})} \subseteq \ldots \subseteq M_{(K,\,\overline{N}_{n,K})} \subseteq \ldots \tag{8.4}$$

is stationary, we conclude that the sequence (8.3) is stationary as well. We conclude that $\overline{M(K)}$ satisfies (ACC) on the set of all W_GK-invariant left-$\overline{A(K)}$-submodules. From theorem (A1.14), we conclude that $\overline{M(K)}$ is a left-noetherian $\overline{A(K)}$-module. $\triangle$

(8.5) THEOREM.

M is totally noetherian (totally artinian) if and only if, for all $H \in S(G)$, $M(H)$ is a left-noetherian (left-artinian) $A(G)$-module via the restriction map r^G_H.

PROOF. We treat only the M totally noetherian case.

Assume first that M is totally noetherian, and let $H \in S(G)$. From proposition (8.4) *(1)*, we conclude that $M_{G/H}(G) = M(H)$ is a left-noetherian $A(G)$-module.

Assume now that, for all $H \in S(G)$, $M(H)$ is a left-noetherian $A(G)$-module via the restriction map r^G_H. If $X \in$ *G-Set*, it follows easily that M_X satisfies the same property. Hence it is enough to show that M is noetherian.

Since $M(H)$ is a left-noetherian $A(G)$-module, we conclude easily that $M(H)$ is a left-noetherian $A(H)$-module as well. Let

$$M_1 \subseteq M_2 \subseteq \ldots. \subseteq M_n \subseteq \ldots \tag{8.5}$$

be an ascending chain of submodules of M. If we evaluate these submodules at $H \in S(G)$, we obtain a chain

$$M_1(H) \subseteq M_2(H) \subseteq .... \subseteq M_n(H) \subseteq ... \tag{8.6}$$

of left $A(H)$ submodules of $M(H)$. Since $M(H)$ is a left-noetherian $A(H)$-module, we conclude that there exists $n_H \geq 1$, such that $M_m(H) = M_{n_H}(H)$, for $m \geq n_H$. If $n = \max\ (n_H \mid H \in S(G))$, we conclude that $M_m(H) = M_n(H)$, for all $m \geq n$ and for all $H \in S(G)$. Hence the sequence (8.5) is stationary. $\triangle$

The following result is a consequence of the argument used in the proof of the theorem (8.5).

(8.6) COROLLARY.

Assume that M is a left-A-module such that $M(H)$ is a left-noetherian (left-artinian) $A(H)$-module, for all $H \in S(G)$. Then M is noetherian (artinian).

The theorem (8.5) has the following immediate corollaries.

(8.7) COROLLARY.

The following statements are equivalent:

(1) A is a totally left-noetherian (totally left-artinian) Green functor.

(2) $A(G)$ is a left-noetherian (respectively left-artinian) ring, and $A(H)$ is finitely generated over $A(G)$ for all $H \in S(G)$.

(3) $A(G)$ is a left-noetherian (respectively left-artinian) ring, and $\overline{A(K)}$ is finitely generated over $A(G)$ for all $K \in \mathcal{P}(A)$.

If A is commutative, then the above conditions are also equivalent to the following:

(4) For all $H \in S(G)$, $A(H)$ is finitely generated over $A(G)$, and, for some $H \in S(G)$, $A(H)$ is noetherian (respectively artinian).

(5) For all $K \in \mathcal{P}(A)$, $\overline{A(K)}$ is finitely generated over $A(G)$, and, for some $K \in \mathcal{P}(A)$, $\overline{A(P)}$ is noetherian (respectively artinian).

PROOF. (1) $\Leftrightarrow$ (2) is an immediate consequence of theorem (8.5).

It is clear that (2) $\Rightarrow$ (3). We show that (3) $\Rightarrow$ (2). Let $H \in \mathcal{P}(A)$. We show first that $A(H)$ is a finitely generated $A(G)$-module. We use induction over $\mathrm{ht}_A(H)$, the height of H in $\mathcal{P}(A)$. If $\mathrm{ht}_A(H) = 1$, then $H \in \mathrm{Min}\ \mathcal{P}(A)$; hence $A(H) = \overline{A(H)}$ is a finitely generated

$A(G)$-module. Now assume that $H \in \mathcal{P}_i(A)$ for some $i > 1$. The map

$$\sum_{\substack{K<H \\ K\in\mathcal{P}(A)}} t_K^H : \bigoplus_{\substack{K<H \\ K\in\mathcal{P}(A)}} A(K) \longrightarrow Tr_A(H)$$

is a surjection of $A(G)$-modules (by proposition (4.1.6) *(1)*). From the induction hypothesis, it follows that $A(K)$ is a finitely generated $A(G)$-module, for all $K \in \mathcal{P}(A)$ such that $K < H$. From the above surjection it follows that $Tr_A(H)$ is a finitely generated $A(G)$-module as well. From the short exact sequence

$$0 \longrightarrow Tr_A(H) \longrightarrow A(H) \longrightarrow \overline{A(H)} \longrightarrow 0$$

we conclude that $A(H)$ is a finitely generated $A(G)$-module. Finally, assume $H \notin \mathcal{P}(A)$, and suppose that $A(H) \neq 0$. From proposition (4.1.6) *(1)*, we conclude that the map

$$\sum_{\substack{K<H \\ K\in\mathcal{P}(A)}} t_K^H : \bigoplus_{\substack{K<H \\ K\in\mathcal{P}(A)}} A(K) \longrightarrow Tr_A(H) = A(H)$$

is a surjection of $A(G)$-modules. Since $A(K)$ is a finitely generated $A(G)$-module, for all $K \in \mathcal{P}(A)$, it follows that $A(H)$ is a finitely generated $A(G)$-module as well.

When A is commutative, the equivalences (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5), are immediate consequences of theorems (A2.8), and (A2.9). $\triangle$

(8.8) COROLLARY.

If A is a totally left-artinian Green functor, then A is totally left-noetherian.

Let A be a Green functor. It is natural to ask under which conditions A has a totally noetherian left-module.

(8.9) THEOREM.

If M is a totally noetherian left-A-module, then $A/\mathrm{Ann}_A(M)$ is a totally noetherian left-A-module as well.

PROOF. Since M is noetherian we conclude that M is finitely generated. Suppose that $M = A\langle m\rangle$, for some $m \in M(X)$, and some finite G-set X. The map

$$\frac{A}{\mathrm{Ann}_A(M)} \longrightarrow M_X, \qquad a \longmapsto a \times m, \tag{8.7}$$

is well defined, and is an injective morphism of left A modules. Since $A/\mathrm{Ann}_A(M)$ can be identified with a submodule of the totally noetherian module M_X, we conclude that $A/\mathrm{Ann}_A(M)$ is totally noetherian. $\triangle$

We now investigate necessary and sufficient conditions under which a left-A-module M is noetherian (artinian).

(8.10) THEOREM.

M is noetherian (artinian) if and only if, for all $H \in \mathcal{P}(A)$, and all submodules N of M, $\overline{N(H)}$ is a left-noetherian (respectively left-artinian) $A(H)$-module.

PROOF. We treat only the M noetherian case.

Assume that M is noetherian, and let N be a submodule of M. Since N is also noetherian, we can apply proposition (8.4) *(2)* to conclude that $\overline{N(H)}$ is a left-noetherian $\overline{A(H)}$-module (hence a left-noetherian $A(H)$-module as well).

Conversely, assume that $\overline{N(H)}$ is a left-noetherian $A(H)$-module for all $H \in \mathcal{P}(A)$ and all submodules N of M. Let

$$N_1 \subseteq N_2 \subseteq .... \subseteq N_i \subseteq ... \tag{8.8}$$

be an ascending chain of submodules of M. Let N be the submodule $N = \bigcup_{i \geq 1} N_i$. We show that $N = N_m$ for some $m \geq 1$. Let $A_1 = A/\mathrm{Ann}_A(M)$. We proceed by induction over the primordial length of A_1.

Suppose that $pl_{A_1} = 1$. We conclude that all the primordial subgroups of A_1 are minimal. Since $\mathcal{P}(M) \subseteq \mathcal{P}(A_1)$, we conclude that all the primordial subgroups of M are minimal as well. It follows that $\overline{M(H)} = M(H)$ for all $H \in \mathcal{P}(A_1)$. From the hypothesis, we conclude that $M(H)$ is noetherian for all $H \in \mathcal{P}(A_1)$. If we evaluate the sequence (8.8) at $H \in \mathcal{P}(A_1)$, we conclude that $N(H) = N_{k_H}(H)$ for some $k_H \geq 1$. If $k = \max\ (k_H \mid H \in \mathcal{P}(A_1))$, it follows that $N(H) = N_k(H)$ for all $H \in \mathcal{P}(A_1)$. From corollary (5.10), we conclude that $N = N_k$.

Assume now that $pl_{A_1} > 1$. Using the above argument, we choose $k \geq 1$ such that $N(K) = N_k(K)$ for all $K \in \mathrm{Min}\ \mathcal{P}(A_1)$. Notice that, according to proposition (6.2.1), $\mathrm{Min}\ \mathcal{P}(A_1) = \mathrm{Min}\ \mathcal{P}(M)$. Consider the following submodule of M:

$$N' = \sum_{K \in \mathrm{Min} \mathcal{P}(A_1)} A\langle N(K)\rangle = \sum_{K \in \mathrm{Min} \mathcal{P}(A_1)} \sum_{n \in N(K)} A\langle n \rangle. \tag{8.9}$$

We set $M' = M/N'$. We first show that M' satisfies the property from the hypothesis of the theorem. Let S' be a submodule of M'. Then $S' = S/N'$ for some submodule S of M containing N'. In particular

$$S(K) \supseteq N'(K) = N(K), \qquad \text{for all } K \in \mathrm{Min}\ \mathcal{P}(A_1), \tag{8.10}$$

therefore

$$\overline{S'(K)} = \frac{S(K)}{N(K)} \subseteq \frac{M(K)}{N(K)} \tag{8.11}$$

is a left-noetherian $A(K)$-module for all $K \in \text{Min}\ \mathcal{P}(A_1)$.

Now let $H \in \mathcal{P}(A)$, and assume H contains properly some minimal primordial subgroup of M. Then

$$Tr_S(H) = \sum_{L<H} t_L^H\ S(K) \supseteq \sum_{\substack{K \in \text{Min}\mathcal{P}(A_1) \\ K<H}} t_K^H\ S(K) \supseteq \sum_{\substack{K \in \text{Min}\mathcal{P}(A_1) \\ K<H}} t_K^H\ N(K) = N'(H). \quad (8.12)$$

From formula (8.12), we conclude that

$$\overline{S'(H)} = \frac{S'(H)}{Tr_{S'}(H)} \cong \frac{S(H)}{Tr_S(H)} = \overline{S(H)}. \quad (8.13)$$

From formulae (8.11) and (8.13), we conclude that $\overline{S'(H)}$ is a left-noetherian $A(H)$-module for all $H \in \mathcal{P}(A)$. In particular, N/N' satisfies the hypotheses of the theorem. Since $\mathcal{P}(N/N') \subseteq \mathcal{P}(A_1)$, but $(N/N')(K) = 0$ for all $K \in \text{Min}\ \mathcal{P}(A_1)$, we use proposition (6.2.1) to conclude that the primordial length of $A/\text{Ann}_A(N/N')$ is smaller than pl_{A_1}. From the induction hypothesis, it follows that N/N' is noetherian. In particular, the ascending chain

$$\frac{N_k}{N'} \subseteq \frac{N_{k+1}}{N'} \subseteq \ldots \subseteq \frac{N_n}{N'} \subseteq \ldots \quad (8.14)$$

is stationary. We conclude that $N/N' = N_m/N'$, for some $m \geq k$. Hence $N = N_m$. $\triangle$

The theorem (8.10) has the following corollary.

(8.11) COROLLARY.

If A is a left-artinian Green functor, then A is left-noetherian.

PROOF. From proposition (8.4) *(2)*, we conclude that $\overline{A(H)}$ is left-artinian, hence left-noetherian, for all $H \in \mathcal{P}(A)$. Let N be a submodule of A. Then $\overline{N(H)}$ is a left-artinian $A(H)$-module. Since $\overline{N(H)}$ is, in fact, a left-$\overline{A(H)}$-module, we conclude that $\overline{N(H)}$ is a left-artinian $\overline{A(H)}$-module. It follows that $\overline{N(H)}$ is a left-noetherian $\overline{A(H)}$-module, hence a left-noetherian $A(H)$-module. The corollary follows now from theorem (8.10). $\triangle$

Unfortunately, the noetherian A-modules do not behave so nicely as the totally noetherian ones do. In particular, theorem (8.9) is no longer true if the assumption totally noetherian is weakened to noetherian.

(8.12) EXAMPLE. Let p be a prime number, and let $G = \mathbf{Z}_p$. Let $A(G)$ be a non-noetherian domain of characteristic p. We may choose, for example, $A(G) = \mathbf{Z}_p[X_i]_{i \geq 1}$, the polynomial ring over $\mathbf{Z}_p$ in countably many indeterminates. Let $A(1)$ be the field of fractions of $A(G)$. A is a Green functor with the restriction map r_1^G being the inclusion $A(G) \longrightarrow A(1)$, and the transfer map $t_1^G = 0$. Let M be the A-module given by $M(G) = 0$, and $M(1) = A(1)$. Then M is simple, hence noetherian. Since M is 1-characteristic, and $\mathrm{Ann}_{A(1)}(M(1)) = 0$, we use formula (6.19) to conclude that $\mathrm{Ann}_A(M) = 0$. However, A is not noetherian.

A left-noetherian Green functor is not necessarily totally left-noetherian as is shown by the following example.

(8.13) EXAMPLE. Again let $G = \mathbf{Z}_p$, where p is some prime number. Let $A(G) = \mathbf{Z}_p$, and $A(1) = \mathbf{Z}_p[X]$. Then A is a Green functor with the restriction map r_1^G being the inclusion $A(G) \longrightarrow A(1)$, and the transfer map $t_1^G = 0$. From corollary (8.6), it follows that A is noetherian. However, $A_{G/1}(G) = A(1)$ is not a finitely generated $A(G)$-module via the restriction map r_1^G. From corollary (8.7), we conclude that A is not totally noetherian.

(8.14) DEFINITION. Let A be a Green functor and let P be an ideal of A. Then P is called a *prime ideal* of A if, whenever I and J are ideals of A such that $I \cdot J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

When A is a commutative Green functor, we can give an equivalent condition for an ideal P of A to be prime.

(8.15) LEMMA.

If A is a commutative Green functor, then P is a prime ideal of A if and only if, whenever $H,\ K \in S(G)$ and $x \in A(H)$, $y \in A(K)$ are such that $x \times y \in P(G/H \times G/K)$, then $x \in P(H)$ or $y \in P(K)$.

PROOF. Immediate consequence of theorem (5.12). $\triangle$

(8.16) THEOREM (Cohen's Theorem).

Assume that A is a commutative Green functor such that all the prime ideals of A are finitely generated. Then A is noetherian.

PROOF. Write Γ for the set of ideals of A which are not finitely generated. If $\Gamma \neq \emptyset$, then, by Zorn's lemma, Γ contains a maximal element I. Then I is not a prime ideal. Hence, according to lemma (8.15), there are elements $x \in A(H)$, $y \in A(K)$, such that $x \notin I(H)$, $y \notin I(K)$, but $x \times y \in I(G/H \times G/K)$. Now $I + A\langle y\rangle$ is bigger than I, hence it is finitely generated. Choose X a finite G-set, and $u \in I(X)$, such that

$$I + A\langle y\rangle = A\langle u\rangle + A\langle y\rangle.$$

Let $(I:y)$ be the maximal ideal J of A, such that $J \cdot A\langle y\rangle \subseteq I$. Notice that

$$(I:y)(Z) = \{a \in A(Z) \mid a \times y \in I(Z \times G/K)\}, \qquad \text{for all } Z \in G-Set. \tag{8.15}$$

It is clear that $I \subsetneq (I:y)$, and $x \in (I:y)(H)$. Since $(I:y)$ is stricly bigger then I, we conclude that $(I:y)$ has a finite system of generators. Assume that $(I:y) = A\langle v\rangle$, for some finite G-set Y, and some $v \in (I:y)(Y)$. It is easy to check that

$$I = A\langle u\rangle + A\langle v \times y\rangle.$$

Hence $I \notin \Gamma$, which is a contradiction. Therefore $\Gamma = \emptyset$. Hence, according to proposition (8.3) *(4)*, A is noetherian. $\triangle$

We now investigate left-artinian Green functors.

(8.17) THEOREM.

If A is a left-artinian Green functor, then $Jac(A)$ is nilpotent.

PROOF. The descending chain

$$Jac(A) \supset \left(Jac(A)\right)^2 \supset \ldots \supset \left(Jac(A)\right)^n \supset \ldots$$

is stationary; hence $\left(Jac(A)\right)^n = \left(Jac(A)\right)^{n+1}$ for some $n \geq 1$. According to corollary (8.11), A is left-noetherian. In particular, $\left(Jac(A)\right)^n$ is finitely generated. From the Nakayama Lemma (7.3.5), it follows that $\left(Jac(A)\right)^n = 0$. $\triangle$

(8.18) COROLLARY.

If A is a commutative artinian Green functor, then $Jac(A) = Nil(A)$.

PROOF. From theorem (8.17), we conclude that $Jac(A) \subseteq Nil(A)$. The reverse containment is given by proposition (7.3.2). $\triangle$

(8.19) THEOREM.

Let A be a left-artinian Green functor. Then there are finitely many isomorphism classes of simple left-A-modules.

PROOF. Let $H \in \mathcal{P}(A)$. We show that there are only finitely many isomorphism classes of simple left-A-modules which are H-characteristic. Let M be such a module. From theorem (7.1.3), we conclude that M is uniquely determined by the $W_G H$-equivariant simple left-$\overline{A(H)}$-module $M(H)$. However, every such module is, in a canonical way, a simple left-module over $T^{\frown}\overline{A(H)}[W_GH]$, the twisted group algebra of W_GH with coefficients in $\overline{A(H)}$ (see definition (A1.1)). Since $\overline{A(H)}$ is left-artinian and $T^{\frown}\overline{A(H)}[W_GH]$ is a finitely generated $\overline{A(H)}$-module, we conclude that $T^{\frown}\overline{A(H)}[W_GH]$ is left-artinian as well. In conclusion, there are only finitely many isomorphism classes of simple left-modules over the twisted group algebra $T^{\frown}\overline{A(H)}[W_GH]$. $\triangle$

In the remainder of this chapter, we prove a structure theorem for commutative artinian Green functors whose Jacobson radical is zero. We begin with some preliminary results.

(8.20) LEMMA.

Let A be a Green functor, and assume $H \notin \mathcal{P}(A)$. Assume that M is a left-A-module such that, for all $K \in \mathcal{P}(A) \cap S(H)$, $M(K)$ is a left-noetherian (left-artinian) $A(K)$-module. Then $M(H)$ is a left-noetherian (respectively left-artinian) $A(H)$-module.

PROOF. We treat only the M noetherian case.

Let

$$N_{1,H} \subseteq N_{2,H} \subseteq \dots \subseteq N_{i,H} \subseteq \dots \tag{8.16}$$

be an ascending chain of submodules of $M(H)$, and let $N_H = \bigcup_{i \geq 1} N_{i,H}$. For $K \in \mathcal{P}(A) \cap S(H)$, consider the following submodules of $M(K)$:

$$N_{i,K} = A(K)\ r_K^H(N_{i,H}), \qquad \text{for } i \geq 1, \tag{8.17}$$

and

$$N_K = A(K)\ r_K^H(N_H). \tag{8.18}$$

Since $M(K)$ is left-noetherian, the chain

$$N_{1,K} \subseteq N_{2,K} \subseteq \dots \subseteq N_{i,K} \subseteq \dots$$

is stationary. We conclude that $N_K = N_{n_K,K}$, for some $n_K \geq 1$. If $n = \max\ (n_K \mid K \in \mathcal{P}(A) \cap S(H))$, then $N_K = N_{n,K}$, for all $K \in \mathcal{P}(A) \cap S(H)$. We show that $N_H = N_{n,H}$. Let $m \in N_H$. Since $H \notin \mathcal{P}(A)$, proposition (4.1.6) *(1)* implies that

$$1_{A(H)} = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H\ a_K, \qquad \text{for some } a_K \in A(K). \tag{8.19}$$

It follows, by multiplying equation (8.19) by m and using the Frobenius axiom, that

$$m = \sum_{K \in \mathcal{P}(A) \cap S(H)} t_K^H\ (a_K r_K^H(m)). \tag{8.20}$$

Since $a_K r^H_K(m) \in N_K = N_{n,K}$, we conclude that

$$a_K r^H_K(m) = \sum_{1 \le j \le j_K} b_{j,K}\, r^H_K(m_{j,K}), \qquad \text{for some } b_{j,K} \in A(K), \text{ and } m_{j,K} \in N_{n,H}. \quad (8.21)$$

Substituting these relations into (8.20), and applying the Frobenius axiom, we obtain that

$$m = \sum_{K \in \mathcal{P}(A) \cap S(H)} \sum_{1 \le j \le j_K} t^H_K(b_{j,K})\, m_{j,K} \in N_{n,H}. \quad (8.22)$$

From (8.22), we conclude that $N_H \subseteq N_{n,H}$. Hence $N_H = N_{n,H}$. $\triangle$

(8.21) COROLLARY.

If $M(K)$ is a left-noetherian (left-artinian) $A(K)$-module for all $K \in \mathcal{P}(A)$, then M is noetherian (respectively artinian).

PROOF. From lemma (8.20), we conclude that $M(H)$ is a left-noetherian (left-artinian) $A(H)$ module for all $H \in S(G)$. Now the result follows from corollary (8.6). $\triangle$

(8.22) THEOREM.

(1) Let A be a Green functor. If $A(H)$ is semisimple artinian for all $H \in S(G)$, then A is an artinian Green functor, and $Jac(A) = 0$.

(2), Conversely, suppose that A is a commutative artinian Green functor and $Jac(A) = 0$. Then $A(H)$ is semisimple artinian, for all $H \in S(G)$.

PROOF. *(1)* From corollary (8.6), we conclude that A is left-artinian. Let $H \in S(G)$. Since $A(H)$ is semisimple it follows, by corollary (7.3.7), that $\big(Jac(A)\big)(H) = 0$. Hence $Jac(A) = 0$.

(2) Since $Jac(A) = Nil(A)$, we conclude that the rings $A(H)$ have trivial nilradicals. Hence it is enough to show that $A(H)$ is artinian for all $H \in S(G)$. Since the proof is rather complicated, we will split it into three major steps. We proceed by induction over pl_A.

STEP I. $pl_A = 1$.

Since $\mathcal{P}(A) = \text{Min } \mathcal{P}(A)$, we conclude that $\overline{A(K)} = A(K)$ for all $K \in \mathcal{P}(A)$. From proposition (8.4) *(2)*, we conclude that $A(K)$ is artinian for all $K \in \mathcal{P}(A)$. From corollary (8.21), it follows that $A(H)$ is artinian for all $H \in S(G)$.

From now on, we assume that $pl_A = m > 1$, and that *(2)* holds for all commutative artinian Green functors (with trivial Jacobson radical) of primordial length $< m$.

STEP II. *It is enough to prove (2) when A is K-characteristic and $A(K)$ is a W_GK-field.*

Assume that the theorem is true for the Green functors satisfying the above condition. If A is a Green functor satifying condition *(2)*, then, by proposition (7.2.2),

$$0 = Jac(A) = \bigcap_{K \in \mathcal{P}(A)} \; \bigcap_{\overline{m} \in \mathrm{Max}\left(\overline{A(K)}\right)} A_{(K,\ \mathrm{eq}(\overline{m}))}. \tag{8.23}$$

Since $\overline{A(K)}$ is artinian for all $K \in \mathcal{P}(A)$, we conclude that there are only finitely many ideals of the form $A_{(K,\ \mathrm{eq}(\overline{m}))}$ in the above intersection (8.23). Let I_1, ... I_n, be all the ideals of the form $A_{(K,\ \mathrm{eq}(\overline{m}))}$ for $K \in \mathcal{P}(A)$, and $\overline{m} \in \mathrm{Max}\left(\overline{A(K)}\right)$.

Fix $i \in \{1,\ 2,\ ...,\ n\}$ and let $I_i = A_{(K,\ \mathrm{eq}(\overline{m}))}$. The Green functor A/I_i is artinian. Its Jacobson radical is zero (because $I_i = A_{(K,\ \mathrm{eq}(\overline{m}))}$ is a radical ideal of A). Moreover, this Green functor is obviously K-characteristic and $A(K)/I_i(K) = \overline{A(K)}/\mathrm{eq}(\overline{m})$ is a W_GK-field. It follows that A/I_i satisfies the hypothesis from Step II. We conclude that $A(H)/I_i(H)$ is artinian, for all $H \in S(G)$. We use formula (8.23), and corollary (A2.2), to conclude that

$$A(H) = \frac{A(H)}{(Jac(A))(H)} = \frac{A(H)}{\bigcap_{1 \le i \le n} I_i(H)}$$

is artinian.

From now on, A will be a Green functor satisfying the condition *(2)* and the hypothesis from Step II. Let I be the ideal

$$I = A\langle A(K)\rangle = \sum_{a \in A(K)} A\langle a\rangle. \tag{8.24}$$

Consider the Green functor $A/\sqrt{I}$. Since $(A/\sqrt{I})(K) = A(K)/A(K) = 0$, and since $\mathrm{Min}\ \mathcal{P}(A) = [K]$, we conclude that the primordial length of $A/\sqrt{I}$ is strictly less than m. According to the induction hypothesis and theorem 5.5,

$$(A/\sqrt{I})(H) = A(H)/(\sqrt{I})(H) = A(H)/\sqrt{I(H)}$$

is artinian for all $H \in \mathcal{P}(A)$. If $\sqrt{I(H)}$ is an artinian ideal of $A(H)$ for all $H \in \mathcal{P}(A)$, then, from the exact sequence

$$0 \longrightarrow \sqrt{I(H)} \longrightarrow A(H) \longrightarrow \frac{A(H)}{\sqrt{I(H)}} \longrightarrow 0,$$

it follows that $A(H)$ is artinian for all $H \in \mathcal{P}(A)$. From lemma (8.20), it follows that $A(H)$ is artinian for all $H \in S(G)$.

The above argument shows that the proof ends once we establish

STEP III. *$\sqrt{I(H)}$ is an artinian ideal of $A(H)$ for all $H \in \mathcal{P}(A)$.*

From formulae (8.24) and (5.6), it follows that

$$I(H) = \sum_{g \in D(H,K)} t^H_{{}^gK}\ A({}^gK). \tag{8.25}$$

We use the Mackey axiom and the fact that Min $\mathcal{P}(A) = [K]$ to conclude that

$$r^H_{{}^lK} t^H_{{}^gK}(a) = 0, \qquad \text{for all } g \neq l \in D(H,K) \text{ and } a \in A({}^gK). \tag{8.26}$$

Now let $g \in D(H,K)$, and let $a \in A({}^gK)$. We use formula (8.26), the Mackey axiom, and the fact that A is K-characterisitic (proposition (6.1.2)) to conclude that

$$r^H_{{}^gK} t^H_{{}^gK}(a) = 0 \quad \Leftrightarrow \quad \sum_{h \in W_H{}^gK} c_h(a) = 0 \quad \Leftrightarrow \quad t^H_{{}^gK}(a) = 0. \tag{8.27}$$

Let

$$S = \{g \in D(H,K) \mid \sum_{h \in W_H{}^gK} c_h(a) \neq 0, \text{ for some } a \in A({}^gK)\}. \tag{8.28}$$

From formulae (8.25) and (8.27), it follows that

$$I(H) = \sum_{g \in S} t^H_{{}^gK}\ A({}^gK).$$

If $S = \emptyset$, we conclude that $I(H) = 0$. Hence $\sqrt{I(H)} = Nil(A(H)) = 0$, because $A(H)$ has trivial nilradical. Therefore, $\sqrt{I(H)}$ is artinian in this case.

Now assume $S \neq \emptyset$. For $g \in S$, let

$$Tr^{W_H{}^gK} = \left\{ \sum_{h \in W_H{}^gK} c_h(a) \mid \text{for } a \in A({}^gK) \right\}, \tag{8.29}$$

and

$${}^gF = A({}^gK)^{W_G{}^gK}. \tag{8.30}$$

Notice that

$$Tr^{W_H{}^gK} = \left\{ r^H_{{}^gK} t^H_{{}^gK}(a) \mid \text{for } a \in A({}^gK) \right\} \subseteq r^H_{{}^gK}(A(H)). \tag{8.31}$$

Since $A({}^gK)$ is a $W_G({}^gK)$-field, we use proposition (A1.21), to conclude that ${}^gF$ is a field, and that $A({}^gK)$ is a finitely generated ${}^gF$ vector space. Since $Tr^{W_H{}^gK}$ is a non-zero ${}^gF$ subspace of $A({}^gK)$, we use formula (8.31) to conclude that $A({}^gK)$ is a finitely generated $A(H)$-module. In particular,

$$_gA(H) = \frac{A(H)}{\text{Ker } r^H_{{}^gK}} \tag{8.32}$$

can be identified with a subring of $A({}^gK)$, and $A({}^gK)$ is finitely generated over $_gA(H)$. From theorem (A2.9), it follows that $_gA(H)$ is artinian. From corollary (A2.2), we conclude that

$$\frac{A(H)}{\bigcap_{g\in S} \text{Ker } r^H_{{}^gK}} \tag{8.33}$$

is artinian.

Now we show that

$$\sqrt{I(H)} \bigcap \Big(\bigcap_{g\in S} \text{Ker } r^H_{{}^gK} \Big) = 0. \tag{8.34}$$

Let $a \in A(H)$ and $n \geq 1$ such that

$$a^n = \sum_{g\in S} t^H_{{}^gK}(a_g), \qquad \text{for some } a_g \in A({}^gK), \tag{8.35}$$

and assume that $r^H_{{}^gK}(a) = 0$ for all $g \in S$. Then $r^H_{{}^gK}(a^n) = 0$ for all $g \in S$. If we restrict equation (8.35) to ${}^gK$, for $g \in S$, we obtain

$$0 = r^H_{{}^gK}(a^n) = r^H_{{}^gK} t^H_{{}^gK}(a_g) + \sum_{\substack{l\in S \\ l\neq g}} r^H_{{}^gK} t^H_{{}^lK}(a_l).$$

From formulae (8.26) and (8.27), it follows that $t^H_{{}^gK}(a_g) = 0$. Since this holds for all $g \in S$, we conclude, from equation (8.35), that $a^n = 0$. Hence $a = 0$, because $A(H)$ has trivial nilradical.

From (8.34), it follows that

$$\sqrt{I(H)} \cong \frac{\sqrt{I(H)} + \bigcap_{g\in S} \text{Ker } r^H_{{}^gK}}{\bigcap_{g\in S} \text{Ker } r^H_{{}^gK}} \leq \frac{A(H)}{\bigcap_{g\in S} \text{Ker } r^H_{{}^gK}} \tag{8.36}$$

Since the ring given by formula (8.33) is artinian, we conclude that $\sqrt{I(H)}$ is an artinian ideal of $A(H)$. $\triangle$

(8.23) COROLLARY.

If A is a commutative artinian Green functor such that $Jac(A) = 0$, then $Jac(\overline{A(H)}) = 0$ for all $H \in \mathcal{P}(A)$.

PROOF. Let $H \in \mathcal{P}(A)$. Since $A(H)$ is semisimple artinian, and $\overline{A(H)}$ is an epimorphic image of $A(H)$, we conclude that $\overline{A(H)}$ is semisimple artinian as well. $\triangle$

The converse of the corollary (8.23) does not hold in general, as is shown by the following example.

(8.24) EXAMPLE. Let p be a prime number such that $p \mid |G|$, and let F be a field of characteristic p. The Burnside ring Green functor B of G (over F) has nontrivial nilradical, but $\overline{B(H)} = F$ for all $H \in S(G)$.

(8.25) COROLLARY.

If A is a commutative artinian Green functor then $(Jac(A))(H) = Jac(A(H))$ for all $H \in S(G)$.

PROOF. Since the Jacobson radical of the Green functor $A/Jac(A)$ is trivial, we use theorem (8.22) to conclude that $A(H)/(Jac(A))(H)$ is semisimple artinian, for all $H \in S(G)$. In particular $(Jac(A))(H) \supseteq Jac(A(H))$ for $H \in S(G)$. The reverse containment follows from theorem (7.3.7). $\triangle$

If A is a commutative artinian Green functor, then A need not be semisimple even if $Jac(A) = 0$. The following example has been pointed out to us by Lewis.

(8.26) EXAMPLE. Let p be a prime number, and let $G = \mathbf{Z}_p$. Consider the following Mackey functors for G (over $\mathbf{Z}_p$):

$$A(H) = \mathbf{Z}_p, \qquad \text{for all } H \in S(G),$$

$$M(H) = \begin{cases} 0\ , & \text{if } H = G, \\ \mathbf{Z}_p, & \text{if } H = 1. \end{cases}$$

The restriction maps $(r_1^G)_A$, $(r_1^G)_M$, corresponding to the functors A and M are the identity of $\mathbf{Z}_p$ and the zero map, respectively. Both transfer maps $(t_1^G)_A$, and $(t_1^G)_M$ are zero. Then A is a Green functor, M is an ideal of A, and

$$(A/M)(H) = \begin{cases} \mathbf{Z}_p, & \text{if } H = G, \\ 0\ , & \text{if } H = 1. \end{cases}$$

Moreover, $(r_1^G)_{A/M}$ is the zero map. If the sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow A/M \longrightarrow 0$$

were split in the category of A-modules, then $(r_1^G)_A$ would equal $(r_1^G)_M + (r_1^G)_{A/M}$. However, both $(r_1^G)_M$ and $(r_1^G)_{A/M}$ are zero, but $(r_1^G)_A$ is the identity of $\mathbf{Z}_p$.

We end this chapter with a few examples of noetherian and artinian Green functors.

(8.27) EXAMPLE. Let R be a noetherian (artinian) ring, and let B be the Burnside ring Green functor for G (over R). Notice that $B(G)$ is a finitely generated R-module. Moreover, $B(H)$ is a finitely generated $B(G)$ module, for all $H \in S(G)$. Hence, it follows, by theorem (8.5), that B is totally noetherian (respectively totally artinian). Similar considerations apply to the Green functor $R_{\mathbf{C}} \otimes R$.

(8.28) EXAMPLE. Assume that $|G|$ is invertible in R, and let A be a left-noetherian (left-artinian) G-algebra over R. From theorem (A1.14) *(2)*, we conclude that A^H is left-noetherian (respectively left-artinian) for all $H \in S(G)$. From corollary (8.6), we conclude that FP_A is a left-noetherian (respectively left-artinian) Green functor.

(8.29) EXAMPLE. Let $H \in S(G)$, and assume that $|W_G H|$ is invertible in R. Let S be a left-noetherian (left-artinian) $W_G H$-algebra over R. From the formula (4.20), theorem (A1.14) *(2)*, and corollary (8.6), we conclude that $J_{G/H}(S)$ is a left-noetherian (left-artinian) Green functor.

(8.30) EXAMPLE. Assume that $|G|$ is invertible in R. Let A be a Green functor such that $\overline{A(H)}$ is left-noetherian (left-artinian) for all $H \in \mathcal{P}(A)$. From corollary (4.2.2) and theorem (4.2.5), it follows that

$$A \cong TA = \prod_{H \in [G \backslash \mathcal{P}(A)]} J_{G/H}(\overline{A(H)}).$$

From the previous example, it follows that A is left-noetherian (respectively left-artinian).

9. Prime and Maximal Ideals.

In this chapter, we investigate the prime ideals of a Green functor A. This chapter has two parts. In 9.1 we present a theorem which describes all the prime ideals of A. When A is commutative, we obtain, as a corollary, that the nilradical of A is the intersection of all prime ideals of A. We also present an induction theorem for prime Green functors. As a corollary, we obtain that a basic property of the prime ideals of the Burnside ring Green functor is, in fact, a common property of the prime ideals of any Green functor. We describe the maximal ideals of a Green functor. We also give characterization theorems for prime and simple Green functors. We conclude this section with a couple of examples.

In 9.2 we assume that A is commutative. We define $\mathrm{Spec}(A)$, and we show that it is compact. We also show that it is connected if and only if $A(G)$ has no nontrivial idempotents.

The Krull dimension of A is investigated in chapter 11.

9.1. CHARACTERIZATION THEOREMS.

The definition of a prime ideal of A is given in chapter 8 (see definition (8.14)). We introduce the following notions.

(9.1.1) DEFINITION. Let S be a ring with a finite group G acting on it by ring automorphisms.

(1) An ideal I of S is called *G-maximal* if I is maximal proper G-invariant ideal of S.

(2) An ideal P of S is called *G-prime* if P is a proper G-invariant ideal of S which has the property that, whenever I and J are two G-invariant ideals of S such that $I \cdot J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

The ring S is *G-simple* if the ideal 0 is G-maximal. The ring S is *G-prime* if the ideal 0 is G-prime. When S is commutative and G-simple, we refer to S as being a *G-field*. When S is commutative and G-prime, we refer to S as being a *G-domain*.

If I is an ideal of S we denote $\bigcap_{g\in G} {}^{g}I$ by $\mathrm{eq}(I)$.

We are now ready to describe the prime ideals of a Green functor A.

(9.1.2) THEOREM. (Characterization Theorem for Prime Ideals)

Let A be a Green functor. Then every prime ideal P of A is of the form

$$P = A_{(H,\,\overline{P_H})}, \tag{9.1}$$

where $H \in \mathcal{P}(A)$, and $\overline{P_H}$ is a W_GH-prime ideal of the ring $\overline{A(H)}$.

Moreover, if $H \in \mathcal{P}(A)$ and $\overline{P_H}$ is a W_GH-prime ideal of $\overline{A(H)}$, then the ideal given by formula (9.1) is prime.

PROOF. Let P be a prime ideal of A, and let $H \in \operatorname{Min} \mathcal{P}(A/P)$. We show that P is H-cocharacteristic. Indeed, let I be the ideal of A containing P such that I/P is the kernel of the map

$$(A/P) \longrightarrow (A/P)_{G/H}. \tag{9.2}$$

Notice that, if $K \in \mathcal{P}(A/P)$, then $[K] \not\lneq [H]$. Let J be the ideal of A containing P such that

$$J/P = \sum_{a \in (A/P)(H)} (A/P)\langle a \rangle.$$

It is clear that P is contained in J. Moreover, from corollary (5.8) and the fact that $(A/P)(L) = 0$ if $[L] < [H]$, it follows that $\mathcal{P}(J/P) = [H]$. From corollary (4.1.13) *(2)*, we conclude that

$$\mathcal{P}\Big(\frac{I}{P} \cdot \frac{J}{P}\Big) \subseteq \mathcal{P}(I/P) \bigcap \mathcal{P}(J/P) = \emptyset$$

In conclusion $I \cdot J \subseteq P$. Since $J \not\subseteq P$, it follows that $I \subseteq P$. We conclude that $I = P$; hence the map (9.2) is injective.

Since P is H-cocharacteristic, it follows, by theorem (6.1.15), that

$$P = A_{(H,\,\overline{P_H})}$$

for some proper W_GH-invariant ideal of $\overline{A(H)}$. We show that $\overline{P_H}$ is a W_GH-prime ideal of $\overline{A(H)}$. Assume that $\overline{I}_H$ and $\overline{J}_H$ are two W_GH-invariant ideals of $\overline{A(H)}$ such that $\overline{I}_H \cdot \overline{J}_H \subseteq \overline{P_H}$. We may suppose that both $\overline{I}_H$ and $\overline{J}_H$ contain $\overline{P_H}$. Let

$$I_1 = A_{(H,\,\overline{I}_H)}, \qquad J_1 = A_{(H,\,\overline{J}_H)}.$$

It is clear that $P \subseteq I_1$, and $P \subseteq J_1$. Moreover, from formula (3.15), it follows easily that

$$\Big(\frac{I_1}{P} \cdot \frac{J_1}{P}\Big)(H) = \frac{I_1(H) \cdot J_1(H)}{P(H)} \cong \frac{\overline{I}_H \cdot \overline{J}_H}{\overline{P_H}} = 0.$$

Since A/P is H-characteristic, we conclude, from proposition (6.1.3) *(1)*, that $(I/P)\cdot(J/P) = 0$; hence $I \cdot J \subseteq P$. Since P is prime, we conclude that $I \subseteq P$, or $J \subseteq P$. If, for example, $I \subseteq P$, then, since the operator $A_{(H,\ -)}$ reflects containments, it follows that $\overline{I}_H \subseteq \overline{P_H}$. We conclude that $\overline{P_H}$ is W_GH-prime.

Now let $H \in \mathcal{P}(A)$, and let $\overline{P}_H$ be an equivariant prime ideal of $\overline{A(H)}$. We show that the ideal

$$P = A_{(H,\ \overline{P}_H)}$$

is prime. Let I and J be two ideals of A such that $I \cdot J \subseteq P$. We may assume that both I and J contain P. It follows, by formula (3.15), that

$$0 = \Big(\frac{I}{P}\cdot\frac{J}{P}\Big)(H) \cong \frac{\overline{I(H)}\cdot\overline{J(H)}}{\overline{P(H)}}.$$

Since $\overline{P_H}$ is W_GH-prime, it follows that $\overline{I(H)} = \overline{P_H}$ or $\overline{J(H)} = \overline{P_H}$. Suppose that $\overline{I(H)} = \overline{P_H}$. We conclude that the ideal I/P of A/P is zero at H. From proposition (6.1.3) *(1)*, it follows that $I = P$. Hence P is prime. $\triangle$

The following corollary is an immediate consequence of theorem (9.1.2), and theorem (A1.22).

(9.1.3) COROLLARY.

(1) If $H \in \mathcal{P}(A)$ and $\overline{p_H}$ is a prime ideal of $\overline{A(H)}$, then

$$P = A_{(H,\ \mathrm{eq}(\overline{p_H}))}$$

is a prime ideal of A.

(2) Suppose that A is commutative, or $\overline{A(H)}$ is left noetherian for all $H \in \mathcal{P}(A)$. Then every prime ideal P of A is of the form

$$P = A_{(H,\ \mathrm{eq}(\overline{p_H}))} \tag{9.3}$$

for some $H \in \mathcal{P}(A)$, and some $\overline{p_H}$ prime ideal of $\overline{A(H)}$.

(9.1.4) COROLLARY.

Suppose that A is a commutative Green functor. Then

(1)

$$Nil(A) = \bigcap_{P \text{ prime ideal}} P \tag{9.4}$$

If $\overline{A(H)}$ is noetherian for all $H \in \mathcal{P}(A)$, then A has only finitely many minimal prime ideals.

(2) Let I be an ideal of A. Then

$$\sqrt{I} = \bigcap_{\substack{I \subseteq P \\ P \text{ prime ideal}}} P \tag{9.5}$$

If A is noetherian, then there are only finitely many minimal prime ideals of A containing I.

PROOF. *(1)* Let $H \in \mathcal{P}(A)$. Let $\overline{Nil_H}$ be the nilradical of $\overline{A(H)}$. Then

$$\overline{Nil_H} = \bigcap_{\overline{p_H} \text{ prime ideal}} \overline{p_H} = \bigcap_{\overline{p_H} \text{ prime ideal}} \mathrm{eq}(\overline{p_H}).$$

From proposition (6.1.21), we conclude that

$$\bigcap_{\overline{p_H} \in \mathrm{Spec}\,(\overline{A(H)})} A_{(H,\, \mathrm{eq}(\overline{p_H}))} = A_{(H,\, \overline{Nil_H})}$$

From corollary (9.1.3) and proposition (6.1.21), it follows that

$$\bigcap_{P \text{ prime ideal}} P = \bigcap_{H \in \mathcal{P}(A)} \bigcap_{\overline{p_H} \in \mathrm{Spec}\,(\overline{A(H)})} A_{(H,\, \mathrm{eq}(\overline{p_H}))} = \bigcap_{H \in \mathcal{P}(A)} A_{(H,\, \overline{Nil_H})} = Nil(A). \tag{9.6}$$

The last equality in formula (9.6) follows from formula (7.19). Now let $H \in \mathcal{P}(A)$. Since the operator $A_{(H,\, -)}$ reflects containments, it follows that every minimal H-cocharacteristic prime ideal of A is of the form $A_{(H,\, \mathrm{eq}(\overline{p_H}))}$, for some minimal prime ideal $\overline{p_H}$ of $\overline{A(H)}$. If $\overline{A(H)}$ is noetherian, then there are only finitely many minimal prime ideals of the ring $\overline{A(H)}$. Hence, if $\overline{A(H)}$ is noetherian for all $H \in \mathcal{P}(A)$, then the Green functor A has only finitely many prime ideals.

(2) Follows immediately from *(1)* applied to the Green functor A/I. $\triangle$

(9.1.5) DEFINITION. A *simple* Green functor A is a Green functor in which 0 is a maximal ideal. A *prime* Green functor A is a Green functor in which the ideal 0 is prime.

A commutative simple Green functor is called a *field*. A commutative prime Green functor is called a *domain*.

Notice that, if A is a prime Green functor, then the ring $A(G)$ has characteristic 0 or a prime number p. We refer to the characteristic of $A(G)$ as the characteristic of A. From theorem (9.1.2), it follows that every prime Green functor has a characteristic subgroup. The following result is an induction theorem for prime Green functors.

(9.1.6) THEOREM.

Let A be a prime Green functor, and let H be the characteristic subgroup of A. Then

(1) If the characteristic of A is $p > 0$, then A is G/H^p-projective.

(2) If the characteristic of A is zero, then every primordial subgroup K of A satisfies $[H] \leq [K] \leq [H^p]$, for some p prime number, where $p \cdot 1_{A(G)}$ is not invertible in $A(G)$.

In both cases, if $K \in \mathcal{P}(A)$ is such that $[H] < [K] \leq [H^p]$, then $\overline{A(K)}$ is annihilated by the integer

$$n = \prod_{g \in D(K,H)} |W_K{}^g H|$$

which is a power of p.

PROOF. Immediate consequence of theorems (6.2.4) and (6.2.5). △

(9.1.7) COROLLARY.

Let P and Q be two prime ideals of a Green functor A, and suppose that $P \subset Q$. Assume that the characteristic subgroups of A/P, and A/Q, are H and K, respectively. Then one of the following holds:

(1) $[H] = [K]$, and $\overline{P(K)} \subset \overline{Q(K)}$.

(2) $[H] < [K] \leq [H^q]$, for some q prime number such that $q \cdot 1_{A(G)}$ is not invertible in $A(G)$. In this case the characteristic of the prime Green functor A/Q is q, and the characteristic of the prime Green functor A/P is either zero or q.

PROOF. If $[H] = [K]$, from the containment

$$P = A_{(K,\ \overline{P(K)})} \subset Q = A_{(K,\ \overline{Q(K)})}$$

it follows that $\overline{P(K)} \subset \overline{Q(K)}$. Suppose now that *(1)* does not hold. From the canonical epimorphism

$$A/P \longrightarrow A/Q \tag{9.7}$$

it follows that $K \in \mathcal{P}(A/P)$. Since A/P is H-characteristic, we use theorem (6.2.5), to conclude that $[H] < [K] \leq [H^q]$, for some prime number q such that $q \cdot 1_{A(G)}$ is not invertible in $A(G)$. Moreover, $\overline{A(K)}$ is annihilated by a power of q. Since $\overline{Q(K)}$ is an W_GK-prime ideal of $\overline{A(K)}$, it follows that $\overline{A(K)}/\overline{Q(K)}$ has characteristic q. From the epimorphism (9.7), it follows that the characteristic of A/P is either zero or q. $\triangle$

It is clear that if P is a maximal ideal of A, then P is prime. Hence, every simple Green functor is a prime Green functor. Therefore a simple Green functor has a characteristic subgroup.

(9.1.8) THEOREM. (Characterization Theorem for Simple Green Functors).

Let A be a Green functor with a characteristic subgroup H. Then A is simple if and only if the $R[W_GH]$ algebra $A(H)$ is both projective and W_GH-simple. In this case $A = J_{G/H}(A(H))$.

PROOF. Assume that A is simple. From theorem (6.1.15), we conclude that $A(H)$ is W_GH-simple (otherwise A will have non-zero ideals which are H-cocharacteristic). Moreover, if $K \in \mathcal{P}(A)$, then $[H] \leq [K]$, and $A_{(K,\,0)}$ is a proper ideal of A. Since the only such ideal is $0 = A_{(H,\,0)}$, it follows that $[K] = [H]$; hence $\mathcal{P}(A) = [H]$. Since A is G/H-projective, and H-characteristic, it follows that A is H-determined; hence $A = J_{G/H}(A(H))$. Since $J_{G/H}(A(H))$ is G/H-projective, we apply lemma (7.4.7), to conclude that the $R[W_GH]$-algebra $A(H)$ is projective.

Suppose now that the $R[W_GH]$ algebra $A(H)$ is both W_GH-simple and projective. From lemma (7.4.7), we conclude that $\mathcal{P}(J_{G/H}(A(H))) = [H]$. We show that $J_{G/H}(A(H))$ is simple. Indeed, let I be a non-zero ideal of $J_{G/H}(A(H))$. We apply proposition (6.1.3) *(1)*, to conclude that $I(H)$ is non-zero. Since $A(H)$ is W_GH- simple, it follows that

$$I(H) = A(H) = J_{G/H}(A(H))(H).$$

Since the only primordial subgroups of $J_{G/H}(A(H))$ are conjugate to H, it follows that $I = J_{G/H}(A(H))$. Finally we show that the injective map

$$j_{G/H} : A \longrightarrow J_{G/H}(A(H))$$

is in fact an isomorphism. Notice that $j_{G/H}(H)$ is surjective. Now the fact that $j_{G/H}$ is surjective follows from corollary (4.1.5), and from the fact that $\mathcal{P}(J_{G/H}(A(H))) = [H]$. $\triangle$

From theorem (9.1.8), it follows that, in order to determine all the H-characteristic simple Green functors for G (over R), it is enough to determine all the $R[W_GH]$ algebras $A(H)$ which are both projective and W_GH-simple.

We now give the following technical result.

(9.1.9) LEMMA.

(1) Let $K \leq H \leq G$ and let M' be an $R[W_GK]$-module. Let $M'\uparrow_{W_HK}^{W_GK}$ be the ordinary induction of modules. Then

$$J_{G/K}(M'\uparrow_{W_GK}^{W_HK}) \;\cong\; (J_{H/K}(M'))\uparrow_H^G.$$

(2) Let $K \leq H \leq G$ be a chain of subgroups of G such that K is normal in both H and G. If $N' \in Mack_R(H/K)$ then

$$\left(\mathrm{Inf}_{H/K}^{H} M'\right)\Big\uparrow_H^G \;\cong\; \mathrm{Inf}_{G/K}^{G}\left(M'\uparrow_{H/K}^{G/K}\right).$$

PROOF. *(1)* Follows by passing to adjoints in the commutative diagram

$$\begin{tikzcd}
Mack_R(G) \arrow[r] \arrow[d, "\downarrow_H^G"] & R[W_GK]\text{-Mod} \arrow[d, "\downarrow_{W_HK}^{W_GK}"] \\
Mack_R(H) \arrow[r] & R[W_HK]\text{-Mod}
\end{tikzcd}$$

where the horizontal arrows are the evaluation functors $M \longmapsto \overline{M(K)}$.

(2) Recall that $\mathrm{Inf}_{G/K}^G$ is right adjoint to $\mathrm{Def}_{G/K}^G$ (see definition (4.2.3)). Now *(2)* follows by passing to adjoins in the commutative diagram

$$\begin{tikzcd}
Mack_R(G) \arrow[r] \arrow[d, "\downarrow_H^G"] & Mack_R(G/K) \arrow[d, "\downarrow_{H/K}^{G/K}"] \\
Mack_R(H) \arrow[r] & Mack_R(H/K)
\end{tikzcd}$$

where the horizontal arrows are the evaluation functors given by formula (4.17). $\triangle$

Let m_H be a maximal ideal of $A(H)$. Since $A(H)$ is W_GH-simple, it follows that $\mathrm{eq}(m_H) = 0$. Let

$$N_G(H, m_H) = \mathrm{Stab}_{N_GH}(m_H) = \{g \in N_GH \mid {}^{g}m_H = m_H\}, \tag{9.8}$$

$$W_G(H, m_H) = N_G(H, m_H)/H. \tag{9.9}$$

Notice that $A(H)/m_H$ is an $R[W_G(H, m_H)]$-algebra. With these notations, we have:

(9.1.10) PROPOSITION.

Let $A(H)$ be W_GH-simple. Let m_H be a maximal ideal of $A(H)$. Then

(1) Consider the $W_G(H, m_H)$-Mackey functor $FP_{A(H)/m_H}$. Then

$$FP_{A(H)} = (FP_{A(H)/m_H})\uparrow^{W_GH}_{W_G(H,m_H)}.$$

(2)

$$J_{G/H}(A(H)) = (\mathrm{Inf}^{N_G(H,m_H)}_{W_G(H,m_H)} FP_{A(H)/m_H})\uparrow^{G}_{N_G(H,m_H)}.$$

(3) The $R[W_GH]$-algebra $A(H)$ is projective if and only if the $R[W_G(H, m_H)]$-algebra $A(H)/m_H$ is projective.

PROOF. *(1)* Notice that if G is a finite group and M' is an $R[G]$-module, then $FP_{M'} = J_{G/1}(M')$. Hence $FP = J_{G/1}$. Moreover, it follows, by theorem (A1.18), that if we regard $A(H)/m_H$ as an $R[W_G(H, m_H)]$-module, then

$$A(H) = (A(H)/m_H)\uparrow^{W_GH}_{W_G(H,m_H)}.$$

Now *(1)* follows from lemma (9.1.9) *(1)*.

(2) It follows, from *(1)* and lemma (9.1.9) *(2)*, that

$$\mathrm{Inf}^{N_GH}_{W_GH} FP_{A(H)} = \mathrm{Inf}^{N_GH}_{W_GH}\left((FP_{A(H)/m_H})\uparrow^{W_GH}_{W_G(H,m_H)}\right) =$$

$$\left(\mathrm{Inf}^{N_G(H,m_H)}_{W_G(H,m_H)} FP_{A(H)/m_H}\right)\Big\uparrow^{N_GH}_{N_G(H,m_H)}.$$

If we apply the functor $\uparrow^{G}_{N_GH}$ to the above equality we obtain

$$\left(\mathrm{Inf}^{N_GH}_{W_GH} FP_{A(H)}\right)\Big\uparrow^{G}_{N_GH} = \left(\mathrm{Inf}^{N_G(H,m_H)}_{W_G(H,m_H)} FP_{A(H)/m_H}\right)\Big\uparrow^{N_GH}_{N_G(H,m_H)}\Big\uparrow^{G}_{N_GH} =$$

$$\left(\mathrm{Inf}^{N_G(H,m_H)}_{W_G(H,m_H)} FP_{A(H)/m_H}\right)\Big\uparrow^{G}_{N_G(H,m_H)}.$$

According to formula (4.20), the left hand side of the above equation is $J_{G/H}(A(H))$.

(3) It follows easily, by proposition (7.4.7), that $A(H)$ is projective if and only if $FP_{A(H)}$ is $W_GH/1$-projective. This happens if and only if $\mathcal{P}(FP_{A(H)}) = [1]$. From *(1)*

and proposition (4.1.9), it follows that the above condition is equivalent to the fact that $FP_{A(H)/m_H}$ is $W_GH/1$-projective. By lemma (7.4.7) again, this last condition is equivalent to the projectivity of the algebra $A(H)/m_H$. $\triangle$

Combining (9.1.8) with (9.1.10), we obtain the following characterization theorem for simple Green functors:

(9.1.11) THEOREM ([T3]).

Let A be a Green functor with a characteristic subgroup H, and let $m \in$ Max $(A(H))$. Then A is simple if and only if

$$A = \left(\mathrm{Inf}_{W_G(H,m)}^{N_G(H,m)} FP_{A(H)/m}\right)\uparrow_{N_G(H,m)}^{G}$$

and the simple $R[W_G(H,m)]$ algebra $A(H)/m$ is projective.

When $A(H)$ is commutative and $m \in$ Max $(A(H))$ there is an easy criterion for determining if the $R[W_G(H,m)]$ algebra $A(H)/m$ is projective. In this case, $A(H)/m$ is a field and $W_G(H,m)$ acts on $A(H)/m$ by field automorphisms. Let $K(H,m) \triangleleft W_G(H,m)$ be the kernel of this action. Let also p be the characteristic of the field A/m, and let $K(H,m,p)$ be a p-Sylow subgroup of $K(H,m)$. With these notations we have:

(9.1.12) PROPOSITION.

Assume that A is an H-characteristic Green functor such that $A(H)$ is W_GH-simple. Let $m \in$ Max$(A(H))$. Then

$$\mathcal{P}(A) \supset \mathrm{Cl}_G\Big(\{L \in S(G) \mid H \triangleleft L \leq K(H,m,p)\}\Big).$$

The above containment is in fact an equality if A is H-determined. In particular, A is a field if and only if p does not divide the order of $K(H,m)$.

PROOF. From proposition (9.1.10) we have

$$J_{G/H}(A(H)) = \left(\mathrm{Inf}_{W_G(H,m)}^{N_G(H,m)} FP_{A(H)/m}\right)\uparrow_{N_G(H,m)}^{G}.$$

It follows, by proposition (4.1.9), that

$$\mathcal{P}(J_{G/H}(A(H)) = \mathrm{Cl}_G\Big(\{L \in S(G) \mid H \triangleleft L, \text{ and } L/H \in \mathcal{P}(FP_{A(H)/m}\}\Big).$$

Let us analyze the primordial subgroups of $FP_{A(H)/m}$. Notice that $FP_{A(H)/m}$ is a 1-characteristic domain of integral characteristic p. By theorem (9.1.6), the primordial subgroups of $FP_{A(H)/m}$ are p-groups. We show that $\mathcal{D}(FP_{A(H)/m}) \subseteq S(K(H,m))$. Indeed, notice that $(A(H)/m)^{W_G(H,m)}$ is a Galois extension with Galois group $W_G(H,m)/K(H,m)$ of the field $A(H)/m = (A(H)/m)^{K(H,m)}$. From classical Galois theory, we know that the relative trace map:

$$tr_{K(H,m)}^{W_G(H,m)} : A(H)/m = (A(H)/m)^{K(H,m)} \longrightarrow (A(H)/m)^{W_G(H,m)}$$

$$tr(a) = \sum_{g \in W_G(H,m)/K(H,m)} g \cdot a$$

is onto. But this map is exactly the transfer map

$$t_{K(H,m)}^{W_G(H,m)} : (FP_{A(H)/m})(K(H,m)) \longrightarrow (FP_{A(H)/m})(W_G(H,m)).$$

From theorem (2.6) and the fact that the primordial subgroups of $FP_{A(H)/m}$ are p-groups, it follows that the primordial subgroups of $FP_{A(H)/m}$ are contained in the p-Sylow subgroups of $K(H,m)$. Now we show that every p subgroup of $K(H,m)$ is primordial. Indeed, if L is a p subgroup of $K(H,m)$, and $L' \subset L$, then L/L' is a p-group. Since $K(H,m)$ represents the kernel of the action of $W_G(H,m)$ on $A(H)/m$, it follows easily that the relative trace map $tr_{L'}^{L} : (A(H)/m)^{L'} \longrightarrow (A(H)/m)^{L}$ will be identically zero. This shows that

$$\mathcal{P}(FP_{A(H)/m}) = \{L \in S(K(H,m)) \mid [L] \leq [K(H,m,p)]\}.$$

It follows immediately that the set of primordial subgroups for $J_{G/H}(A(H))$ is exactly the set which appears in the left side of the containment stated in the proposition (9.1.12). Since $J_{G/H}(A(H))$ is an A-module, it follows that $\mathcal{P}(A)$ contains the primordial subgroups of $J_{G/H}(A(H))$.

Finally, assume that p does not divide the order of $K(H,m)$. From the previous arguments, it follows that the $R[W_G(H,m)]$ algebra $A(H)/m$ is projective. From proposition (9.1.10) *(3)* and theorem (9.1.8), it follows that A is simple.

We now describe the maximal ideals of a Green functor.

(9.1.13) THEOREM (Characterization Theorem for Maximal Ideals)

A prime ideal $P = A_{(H,\ \overline{P_H})}$ is maximal if and only if the following two conditions are satisfied:

(1) $\overline{P_H} = \mathrm{eq}(\overline{m_H})$ for some $\overline{m_H}$ maximal ideal of $\overline{A(H)}$.

(2) The $R[W_G(H,\overline{m_H})]$ algebra $\overline{A(H)}/\overline{m_H}$ is projective.

PROOF. Suppose first that $P = A_{(H,\ \overline{P_H})}$ is maximal. Then $\overline{P_H}$ is an W_GH-maximal ideal of $\overline{A(H)}$. According to theorem (A1.17), it follows that $\overline{P_H} = \mathrm{eq}(\overline{m_H})$ for some $\overline{m_H}$ maximal ideal of $\overline{A(H)}$. Since the Green functor A/P is simple, we conclude, from theorem (9.1.8), that the $R[W_GH]$ algebra

$$(A/P)(H) = A(H)/P(H) \cong \overline{A(H)}/\overline{P_H} = \overline{A(H)}/\mathrm{eq}(\overline{m_H})$$

is projective. From proposition (9.1.10), we conclude that the $R[W_G(H, \overline{m_H})]$ algebra $\overline{A(H)}/\overline{m_H}$ is projective.

Conversely, assume that $\overline{P_H}$ is an ideal of $\overline{A(H)}$ satisfying conditions *(1)* and *(2)* above. From proposition (9.1.10), we conclude that the algebra

$$(A/P)(H) = \overline{A(H)}/\overline{P(H)} = \overline{A(H)}/\mathrm{eq}(\overline{m_H})$$

is both projective and W_GH-simple. It follows, by (9.1.8), that A/P is simple. $\triangle$

We now give some examples.

(9.1.14) EXAMPLE. Let B be the Burnside ring functor for G (over R). Since W_GH acts trivially on $\overline{B(H)}$, we conclude that the prime ideals of B are of the form $B_{(H,\ p)}$, where p is a prime ideal of R. We show that $B_{(H,\ p)}$ is maximal if and only if p is maximal and the characteristic of the residual field R/p does not divide the order of $|W_GH|$. Indeed, condition *(1)* from theorem (9.1.13) is fulfilled if and only if p is maximal. Assume therefore that p is maximal. Since W_GH acts trivially on R/p, the $R[W_GH]$ algebra R/p is projective if and only if $|W_GH|$ is invertible in R/p.

(9.1.15) EXAMPLE. Let $R_{\mathbf{C}}$ be the character ring Green functor. We determine the prime ideals P such that $R_{\mathbf{C}}/P$ has characteristic zero. We claim that the only ideals with this property are the ideals $(R_{\mathbf{C}})_{(H,\ 0)}$, for H cyclic. Indeed, if K is a primordial subgroup of $R_{\mathbf{C}}$, which is not cyclic then $\overline{R_{\mathbf{C}}(K)}$ is finite (example (2.10)). Hence there are no ideals P with the above property which are K-cocharacteristic. If H is cyclic, then $\overline{R_{\mathbf{C}}(H)} = \mathbf{Z}[\zeta_H]$, where ζ_H is a primitive root of unity of order $|H|$. Now the claim follows from the fact that every nonzero prime ideal p of $\mathbf{Z}[\zeta]$ is maximal and its residual field is finite. From corollary (9.1.7), we conclude that all these ideals are minimal.

(9.1.16) Let $R_{\mathbf{C}}\otimes\mathbf{Q}$ be the character ring Green functor for G (over $\mathbf{Q}$). The primordial subgroups of this functor are the cyclic subgroups. Moreover, if H is a cyclic subgroup, then $\overline{(R_{\mathbf{C}} \otimes \mathbf{Q})(H)} = \mathbf{Q}[\zeta_H]$ is a field of characteristic zero. Here ζ_H is again a primitive root of unity of order $|H|$. From corollary (4.2.2) and theorem (4.2.8), we conclude that if we denote the set of cyclic subgroups of G by Cyc, then

$$R_C \otimes \mathbf{Q} \cong T(R_C \otimes \mathbf{Q}) = \bigoplus_{H \in [G \backslash Cyc]} J_{G/H}(\mathbf{Q}[\zeta_H]).$$

It is clear that the Green functors $J_{G/H}(\mathbf{Q}[\zeta_H])$ are simple. We conclude that the only prime ideals of this functor are $(R_C \otimes \mathbf{Q})_{(H,\,0)}$ for H cyclic. They are all maximal. In particular, $R_C \otimes \mathbf{Q}$ is semisimple.

9.2. THE SPECTRUM OF A COMMUTATIVE GREEN FUNCTOR.

Throughout this section, A is a commutative Green functor.

(9.2.1) LEMMA.

If P is a prime ideal of A, then $P(G)$ is a prime ideal of $A(G)$.

PROOF. Assume that a, $b \in A(G)$ are such that $a \cdot b \in P(G)$. From proposition (5.6), it follows that $a \cdot b = a \times b \in P(G) = P(G/G \times G/G)$. From lemma (8.15), we conclude that $a \in P(G)$ or $b \in P(G)$. Δ

(9.2.2) LEMMA.

Let I_G be an ideal of $A(G)$. Let

$$\Psi_{I_G} = \{I \text{ ideal of } A \mid I(G) \subseteq I_G\}. \tag{9.10}$$

Then there exists a uniquely determined ideal ψ_{I_G} which is maximal in Ψ_{I_G}. This ideal satisfies $\psi_{I_G}(G) = I_G$, and

$$J \subseteq \psi_{I_G}, \qquad \text{for all } J \in \Psi_{I_G}. \tag{9.11}$$

If I_G is prime, radical or maximal, then ψ_{I_G} is also prime, radical, or maximal, respectively.

PROOF. Notice that, if $(I_j)_{j\in\Gamma}$ is a family of ideals in Ψ_{I_G}, then $\sum_{j\in\Gamma} I_j \in \Psi_{I_G}$. We set

$$\psi_{I_G} = \sum_{J \in \Psi_{I_G}} J \tag{9.12}$$

It is clear that $\psi_{I_G} \in \Psi_{I_G}$, and that containment (9.11) holds. Let

$$J = A\langle I_G \rangle = \sum_{a \in I_G} A\langle a \rangle, \tag{9.13}$$

be the ideal of A generated by all the elements $a \in I_G$. It follows, by corollary (5.8) *(2)*, that $J(G) = I_G$. This shows that $J \in \Psi_{I_G}$. Hence $J \subseteq \psi_{I_G}$. In particular $I_G = J(G) \subseteq \psi_{I_G}(G)$. Hence $\psi_{I_G}(G) = I_G$.

Assume now that I_G is prime, and let I, J be ideals such that $I \cdot J \subseteq \psi_{I_G}$. From corollary (3.1.4) *(1)*, we conclude that

$$I(G) \cdot J(G) \subseteq (I \cdot J)(G) \subseteq \psi_{I_G}(G) = I_G$$

and since I_G is prime it follows that $I(G) \subseteq I_G$ or $J(G) \subseteq I_G$. Suppose that $I(G) \subseteq I_G$. Then $I \in \Psi_{I_G}$; hence $I \subseteq \psi_{I_G}$.

Suppose now that I_G is radical. By theorem (5.5)

$$(\sqrt{\psi_{I_G}}\,)(G) = \sqrt{\psi_{I_G}(G)} = \sqrt{I_G} = I_G.$$

Hence $\sqrt{\psi_{I_G}} \in \Psi_{I_G}$. This containment implies that ψ_{I_G} is radical.

Finally assume that I_G is maximal, and let J be an ideal of A such that $\psi_{I_G} \subseteq J$. Then $I_G = \psi_{I_G}(G) \subseteq J(G)$. If $I_G = J(G)$, we conclude that $J \in \Psi_{I_G}$; hence $J \subseteq \psi_{I_G}$. In this case $\psi_{I_G} = J$. If $I_G \subset J(G)$ then, from the maximality of I_G, it follows that $J(G) = A(G)$. In particular J contains a trivial unit, hence $J = A$. $\triangle$

(9.2.3) DEFINITION.

The *spectrum* Spec(A) of A, is the set of all prime ideals of A.

The spectrum of A carries a natural Zariski topology which can be defined as follows. Let E be a subset of A. Let

$$V(E) = \{P \in \mathrm{Spec}(A) \mid E \subseteq P\}.$$

One can check easily that the operator V satisfies the following properties:

(1) $V(E) = V(A\langle E\rangle)$.

(2) If $(E_i)_{i \in \Gamma}$ is a collection of subsets of A, then

$$V(\bigcup_{i\in\Gamma} E_i) = V\Big(\sum_{i\in\Gamma} A\langle E_i\rangle\Big) = \bigcap_{i\in\Gamma} V(E_i).$$

(3) If I and J are ideals of A then

$$V(I \cap J) = V(I \cdot J) = V(I) \cup V(J).$$

(4) If I is an ideal of A, then $V(I) = V(\sqrt{I})$.

(5) $V(0) = \mathrm{Spec}(A)$ and $V(A) = \emptyset$.

The Zariski topology of Spec(A) is the topology for which the closed sets are of the form $V(E)$, for $E \subseteq A$. The fact that this is indeed a topology follows from *(2)*, *(3)* and *(5)* above. If $E \subseteq A$, let

$$X(E) = \mathrm{Spec}(A) - V(E)$$

It is clear that $X(E)$ is open, and that every open set is of the above form for some $E \subseteq A$.

(9.2.4) PROPOSITION.

Spec*(A) is compact.*

PROOF. Let $(V_i)_{i \in \Gamma}$ be a family of closed sets of Spec(A) such that $\bigcap V_i = \emptyset$. From *(1)* above, we may assume that $V_i = V(I_i)$ for some ideal I_i of A. Then

$$\emptyset = \bigcap_{i \in \Gamma} V(I_i) = V(\sum_{i \in \Gamma} I_i).$$

It follows that $\sum_{i \in \Gamma} I_i = A$ (otherwise $\sum_{i \in I} I_i$ is contained in a maximal ideal which is prime). In particular,

$$\sum_{i \in \Gamma} I_i(G) = (\sum_{i \in \Gamma} I_i)(G) = A(G).$$

From the above relation, we conclude that there exists a relation of the form

$$\sum_{j=1}^{n} a_{i_j} = 1_{A(G)}, \qquad \text{for some } i_1,\ \ldots,\ i_n \in \Gamma, \text{and } a_{i_j} \in I_{i_j}(G).$$

In particular, the ideal $\sum_{j=1}^{n} I_{i_j}$ contains a trivial unit. Hence $\sum_{j=1}^{n} I_{i_j} = A$. This implies that

$$\emptyset = V(A) = V(\sum_{j=1}^{n} I_{i_j}) = \bigcap_{j=1}^{n} V(I_{i_j}) = \bigcap_{j=1}^{n} V_{i_j}. \qquad \triangle$$

From the lemmas (9.2.1) and (9.2.2), we conclude that there exists two well-defined maps

$$\phi : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A(G)), \qquad \psi : \mathrm{Spec}(A(G)) \longrightarrow \mathrm{Spec}(A)$$

given by $\phi(P) = P(G)$, and $\psi(P_G) = \psi_{P_G}$. The following lemma summarizes the main properties of the maps ϕ and ψ.

(9.2.5) LEMMA.

(1) $\phi \cdot \psi = 1_{\mathrm{Spec}(A(G))}$. *In particular, ϕ is onto.*

(2) ϕ restricts to a bijection

$$\mathrm{Max}\ A \longrightarrow \mathrm{Max}\ A(G)$$

whose inverse is ψ.

(3) If $p_G \in \mathrm{Spec}(A(G))$, then the ideal $\psi(p_G)$ is maximal in the fibre $\phi^{-1}(p_G)$ in the strong sense that, if $P \in \phi^{-1}(p_G)$, then $P \subseteq \psi(p_G)$.

(4) If I_G is an ideal of $A(G)$, then

$$\phi^{-1}(V(I_G)) = V(A\langle I_G\rangle), \qquad \psi(V(I_G)) \subseteq V(\psi_{I_G}).$$

(5) If I is an ideal of A, then

$$\phi(V(I)) = V(I(G)), \qquad \psi^{-1}(V(I)) = V(I(G)).$$

(6) Both ϕ and ψ preserve containment.

(7) Both maps ϕ and ψ are contiuous, and ϕ is closed.

PROOF. *(1)* Obvious.

(2) Follows immediately from lemma (9.2.2) and *(1)*.

(3) Follows immediately from lemma (9.2.2).

(4) Let I_G be an ideal of $A(G)$. We have the following chain of equivalences:

$$P \in \phi^{-1}(V(I_G)) \;\Leftrightarrow \phi(P) \supseteq I_G \;\Leftrightarrow P(G) \supseteq I_G \;\Leftrightarrow P \supseteq A\langle I_G\rangle \;\Leftrightarrow P \in V(A\langle I_G\rangle).$$

This shows that $\phi^{-1}(V(I_G)) = V(A\langle I_G\rangle)$. For the second containment, let $p_G \in V(I_G)$. Then $I_G \subseteq p_G$; hence $\Psi_{I_G} \subseteq \Psi_{p_G}$. In particular, $J \subseteq \psi_{p_G} = \psi(p_G)$, for all $J \in \Psi_{I_G}$. Since $\psi_{I_G} \in \Psi_{I_G}$, we conclude that $\psi_{I_G} \subseteq \psi(p_G)$. Hence $\psi(p_G) \in V(\psi_{I_G})$.

(5) It is clear that, if $P \in V(I)$, then $I \subseteq P$. Thus $I(G) \subseteq P(G) = \phi(P)$. In particular $\phi(P) \in V(I(G))$. For the reverse containment, let $p_G \in V(I(G))$. Then $I(G) \subseteq p_G$, therefore $\Psi_{I(G)} \subseteq \Psi_{p_G}$. In particular, for all $J \in \Psi_{I_G}$, $J \subseteq \psi_{p_G} = \psi(p_G)$. Since $I \in \Psi_{I(G)}$, it follows that $I \subseteq \psi(p_G)$. Therefore $\psi(p_G) \in V(I)$. Now we use *(1)* to conclude that $p_G = \phi(\psi(p_G)) \in \phi(V(I))$, which proves the reverse containment.

The second equality follows immediately because ψ is one-to-one and, by lemma (9.2.2), $(\psi(p_G))(G) = p_G$ for all $p_G \in \mathrm{Spec}(A(G))$.

(6) Obvious.

(7) The continuity of ϕ follows from *(4)*. The fact that ϕ is closed follows from *(5)*. Finally, from *(5)*, we conclude that ψ is continuous. $\triangle$

(9.2.6) COROLLARY.

Spec*(A) is connected if and only if* Spec*(A(G)) is connected.*

PROOF. Suppose that Spec(A) is connected. From the previous lemma, we know that ϕ is continuous and onto. It follows that Spec($A(G)$) is connected as well.

Suppose now that Spec($A(G)$) is connected. Let I and J be two ideals of A such that

$$\mathrm{Spec}(A) = V(I) \cup V(J) \quad \text{and} \quad V(I) \cap V(J) = \emptyset.$$

In particular $\mathrm{Spec}(A) = V(I \cap J)$ and $\emptyset = V(I + J)$. From lemma (9.5), we conclude that

$$\mathrm{Spec}(A(G)) = \phi(\mathrm{Spec}(A)) = \phi(V(I \cap J)) = V(I(G) \cap J(G)) = V(I(G)) \cup V(J(G))$$

and

$$\emptyset = \phi(V(I + J)) = V(I(G) + J(G)) = V(I(G)) \cap V((J(G)).$$

Since Spec($A(G)$) is connected, we conclude that either $V(I(G))$ or $V(J(G))$ is empty. Assume, for example, that $V(I(G)) = \emptyset$. It follows that $I(G) = A(G)$. In particular, the ideal I contains a trivial unit. Therefore $V(I) = \emptyset$. Hence Spec(A) is connected. $\triangle$

It is well known that Spec($A(G)$) is connected if and only if $A(G)$ has no non-trivial idempotents. With this observation, corollary (9.2.6) can be restated as follows:

(9.2.7) COROLLARY.

Spec*(A) is connected if and only if $A(G)$ has no non-trivial idempotents.*

(9.2.8) EXAMPLE. Let R_C be the character ring Green functor for G. From [Se], we know that $\mathrm{Spec}(R_C(G))$ is connected. From corollary (9.2.6), it follows that $\mathrm{Spec}(R_C)$ is also connected.

(9.2.9) EXAMPLE. Let G be a finite group, and let Π be a set of prime numbers. Let $B_{(\Pi)}$ be the Burnside ring Green functor for G (over $\mathbf{Z}_{(\Pi)}$). From [Dr1], we know that $B_{(\Pi)}(G)$ has no non-trivial idempotents if and only if G is solvable and the order of G is divisible only by primes not in Π. Therefore $\mathrm{Spec}(B_{(\Pi)})$ is connected if and only if G is solvable and $|G|_{\Pi} = 1$. For example, $\mathrm{Spec}(B)$ is connected if and only if G is solvable.

10. Localization.

Throughout this chapter, A is a commutative Green functor and M is an A-module. Notice that, since A is a commutative Green functor, A can be regarded as a Green functor for G over $A(G)$. With this convention, an A-module M is also a G-Mackey functor over $A(G)$. Let U be a multiplicative subset of $A(G)$. The *localization of M at U* is the family of $A(G)$-modules $\left(U^{-1}M(H)\right)_{H\in S(G)}$. It was noticed by Thévenaz (see [T3]) that the above collection of $A(G)$-modules is a Mackey functor. We denote it by $U^{-1}M$. We show that $U^{-1}A$ is a Green functor and an A-algebra, and that $U^{-1}M$ is an $U^{-1}A$-module. We show that this localization has the expected universality property. We also show that the functor U^{-1} is exact. We investigate the relation between the primordial subgroups of M and the primordial subgroups of the localizations of M at various maximal ideals of $A(G)$. We also give a structure theorem for the prime ideals of the Green functor $U^{-1}A$. It follows, as a corollary of this result, that if $p_G \in \operatorname{Spec}(A(G))$, then the localization of A at p_G can be regarded as the localization of A at the prime ideal ψ_{p_G} of A. For A a domain we define the fraction functor $F(A)$ of A as the localization of A at the multiplicative subset $A(G)-\{0\}=A(G)^*$ of $A(G)$. We give an example which shows that $F(A)$ need not be a Green field. We conclude with a couple of necessary and sufficient condition for $F(A)$ to be a field.

Let $U(G)$ be a multiplicative subset of $A(G)$, that is, a subset containing $1_{A(G)}$ and closed under multiplication. For each $H \in S(G)$, let $U(H) = r_H^G(U(G))$. Then $U(H)$ is a multiplicative subset of $A(H)$. Since $M(H)$ is an $A(H)$-module, we can construct the localized module $(U^{-1}M)(H) = U(H)^{-1}M(H)$. If $K \leq H$ and if $g \in G$, we define

$$r_K^H : (U^{-1}M)(H) \longrightarrow (U^{-1}M)(K); \qquad r_K^H\left(\frac{m}{r_H^G(u)}\right) = \frac{r_K^H(m)}{r_K^G(u)},$$

$$t_K^H : (U^{-1}M)(K) \longrightarrow (U^{-1}M)(H); \qquad t_K^H\left(\frac{m}{r_K^G(u)}\right) = \frac{t_K^H(m)}{r_H^G(u)},$$

$$c_g : (U^{-1}M)(H) \longrightarrow (U^{-1}M)({}^gH); \qquad c_g\left(\frac{m}{r_H^G(u)}\right) = \frac{c_g(m)}{r_{{}^gH}^G(u)}.$$

(10.1) THEOREM.

(1) The family of modules $U^{-1}M$ with the above data is an A-module, and the family of canonical maps

$$j_M(H) : M(H) \longrightarrow (U^{-1}M)(H); \qquad j_M(H)(m) = \frac{m}{1}$$

is a morphism of A-modules.

(2) $U^{-1}A$ is a Green functor, and the map j_A defined above is a morphism of Green functors.

(3) $U^{-1}M$ is an $U^{-1}A$-module.

PROOF. Straightforward calculations using the axioms 1.1. $\triangle$

(10.2) THEOREM.

$$U^{-1}M \cong M \,\square_A\, U^{-1}A.$$

PROOF. For $H \in S(G)$, let f_H be the map

$$(U^{-1}M)(H) \xrightarrow{f_H} (M \,\square_A\, U^{-1}A)(H), \qquad \frac{m}{r_H^G(u)} \longmapsto m \,\square_H \left(\frac{1}{r_H^G(u)}\right)$$

where

$$m \,\square_H \left(\frac{1}{r_H^G(u)}\right) = m \otimes_{A(H)} \left(\frac{1}{r_H^G(u)}\right) \pmod{I_A(H)}$$

(see theorem (3.1.2) and formula (3.13)). Using formula (3.13), one can immediately check that this map gives a map $f : U^{-1}M \longrightarrow M \,\square_A\, U^{-1}A$. The inverse of f is the map g corresponding to the obvious pairing

$$g_H : M(H) \otimes_{A(H)} (U^{-1}A)(H) \longrightarrow (U^{-1}M)(H), \qquad \left(m, \left(\frac{a}{r_H^G(u)}\right)\right) \longmapsto \frac{am}{r_H^G(u)}. \quad \triangle$$

The Green functor $U^{-1}A$ has the following universality property:

(10.3) PROPOSITION.

Assume that $f : A \longrightarrow A'$ is a morphism of Green functors. Suppose that for all $u \in U(G)$, $f(G)(u)$ is a unit in $A'(G)$. Then there exits a unique map of Green functors f' making the following diagram commute:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j_A"'] & & A' \\ & U^{-1}A \arrow[ur, "f'"'] & \end{tikzcd}$$

PROOF. If $H \in S(G)$, define

$$f'(H) : (U^{-1}A)(H) \longrightarrow A'(H); \qquad f'(H)\Big(\frac{a}{r_H^G(u)}\Big) = r_H^G((f(G)(u))^{-1}) \cdot f(H)(a).$$

It is a straightforward calculation to check that this map well-defined and functorial. It is easy to see that the above f' is the only map making the above diagram commute. $\triangle$

Notice that, if $H \in S(G)$, then $U(H)$ might contain zero divisors even if $U(G)$ doesn't. In particular, $(U^{-1}M)(H)$ can be zero even if $M(H)$ is non-zero. This shows, for example, that the map j_A is not necessarily injective.

(10.4) EXAMPLE. Let p be a prime number, and let $G = \mathbf{Z}_p$. Let $A(G) = \mathbf{Z}$, and $A(1) = \mathbf{Z}_p$. If one defines $r_1^G : \mathbf{Z} \longrightarrow \mathbf{Z}_p$ to be the canonical projection of $\mathbf{Z}$ onto $\mathbf{Z}_p$ and the transfer map t_1^G to be zero, then A becomes a Green functor. Let $U(G)$ be the multiplicative subset of $\mathbf{Z}$ consisting of all powers of p. Then $(U^{-1}A)(G) = \mathbf{Z}[\frac{1}{p}]$, and $(U^{-1}A)(1) = 0$.

Since the localization with respect to the multiplicative subset $U(H)$ of $A(H)$ is an exact functor from the category of $A(H)$-modules to the category of $(U^{-1}A)(H)$-modules we conclude that:

(10.5) PROPOSITION.

The functor $U^{-1} : A\text{-}Mod \longrightarrow (U^{-1}A)\text{-}Mod$ is exact.

Let X be a finite G-set. Then both $U^{-1}(M_X)$ and $(U^{-1}M)_X$ are $U^{-1}A$-modules. We have the following result:

(10.6) PROPOSITION.

The $U^{-1}A$-modules $U^{-1}(M_X)$ and $(U^{-1}M)_X$ are naturally isomorphic.

PROOF. From formula (1.1) and theorem (10.2) it follows that

$$(U^{-1}M)_X = (M \,\Box_A\, U^{-1}A)_X \cong (M_X) \,\Box_A\, U^{-1}A \cong U^{-1}(M_X). \qquad \triangle$$

(10.7) COROLLARY.

(1) If M is H-bounded, then $U^{-1}M$ is H-bounded.

(2) If M is H-characteristic, then $U^{-1}M$ is either zero or H-characteristic.

(3) If M is X-injective, then $U^{-1}M$ is X-injective.

(4) If M satisfies X-injective induction, then $U^{-1}M$ satisfies X-injective induction.

(5) If M is H-determined, then $U^{-1}M$ is either zero or H-determined.

PROOF. *(1)* Since $M(K) = 0$, for $K < H$, it follows that

$$(U^{-1}M)(K) = U(K)^{-1}M(K) = 0 \qquad \text{for } K < H.$$

(2) Since M is H-bounded, it follows that $U^{-1}M$ is H-bounded as well. We use the fact that

$$\theta_M^{G/H} : M \longrightarrow M_{G/H}$$

is injective and the fact that U^{-1} is exact to conclude that

$$U^{-1}(\theta_M^{G/H}) : U^{-1}M \longrightarrow U^{-1}(M_{G/H})$$

is injective. We indentify $U^{-1}(M_{G/H})$ with $(U^{-1}M)_{G/H}$. It is easy to see that after this identification, the map $U^{-1}(\theta_M^{G/H})$ becomes the map $\theta_{U^{-1}M}^{G/H}$. In conclusion, if $U(H)^{-1}M(H) = 0$, then $U^{-1}M$ is zero. Otherwise, $U^{-1}M$ is H-characteristic.

(3) Let ψ be a splitting of the injective map

$$\theta_M^X : M \longrightarrow M_X.$$

Since U^{-1} is exact, we conclude that $U^{-1}(\psi)$ is a splitting for $U^{-1}(\theta_M^X)$. If we identify $U^{-1}(M_X)$ with $(U^{-1}M)_X$, then the map $U^{-1}(\theta_M^X)$ becomes the map $\theta_{U^{-1}M}^X$. Hence $U^{-1}M$ is X-injective.

(4) Follows by a similar argument applied to the exact sequence

$$0 \longrightarrow M \xrightarrow{d^0} M_X \xrightarrow{d^1} M_{X^2}.$$

(5) Follows from *(2)* and *(4)* because M is H-determined if and only if M is H-characteristic and M satisfies G/H-injective induction. $\triangle$

Let now I be an ideal of A. Denote by $I \cdot U^{-1}A$ the ideal of $U^{-1}A$ generated by $j_A(I)$. Notice that $I \cdot U^{-1}A = U^{-1}I$ (this follows, for example, from theorem (10.2)).

(10.8) THEOREM.

(1) All the ideals of $U^{-1}A$ are of the form $I \cdot U^{-1}A$, with I an ideal of A.

(2) Every prime ideal of $U^{-1}A$ is of the form $P \cdot U^{-1}A$, with P a prime ideal of A such that $P(G) \cap U(G) \neq \emptyset$. Conversely, if P is a prime ideal of A such that $P(G)$ is disjoint from $U(G)$, then $P \cdot U^{-1}A$ is a prime ideal of $U^{-1}A$.

(3) Every maximal ideal of $U^{-1}A$ is of the form $\psi_{p_G} \cdot U^{-1}A$ where p_G is a prime ideal of $A(G)$ disjoint from $U(G)$ and maximal with the above property. Conversely, if p_G is a prime ideal of $A(G)$ satisfying the above condition then $\psi_{p_G} \cdot U^{-1}A$ is a maximal ideal of $U^{-1}A$.

PROOF. *(1)* Let J be an ideal of $U^{-1}(A)$. Let $I = j_A^{-1}(J)$. It is clear that $I \cdot U^{-1}(A) \subseteq J$. Conversely, notice that, if $H \in S(G)$ and $a/r_H^G(u) \in J(H)$, for some $a \in A(H)$ and $u \in U(G)$, then

$$\frac{a}{1} = r_H^G(u) \cdot \left(\frac{a}{r_H^G(u)}\right) \in J(H).$$

Hence $a \in I(H)$. This shows that

$$\frac{a}{r_H^G(u)} = \frac{1}{r_H^G(u)} \cdot \frac{a}{1} = \frac{1}{r_H^G(u)} \cdot j_A(H)(a) \in (j_A(I) \cdot U^{-1}A)(H) = (I \cdot U^{-1}A)(H).$$

Hence $J(H) \subseteq (I \cdot U^{-1}A)(H)$.

(2) If Q is a prime ideal of $U^{-1}A$, we set $P = j_A^{-1}(Q)$. Then P is a prime ideal of A. From the above argument it follows that $Q = P \cdot U^{-1}A$. Moreover, since Q does not contain units of $U^{-1}A$, we conclude that $Q(G)$ does not contain units of $(U^{-1}A)(G) = U(G)^{-1}A(G)$. In particular, $P(G)$ is disjoint from $U(G)$. Conversely, assume that P is a prime ideal of A such that $P(G) \cap U(G) = \emptyset$. We now use the fact that $P \cdot U^{-1}A$ can be canonically identified with the ideal $U^{-1}P$ of $U^{-1}A$. From the exactness of the functor U^{-1}, we conclude that

$$U^{-1}\left(\frac{A}{P}\right) = \frac{U^{-1}A}{U^{-1}P}.$$

Assume that the domain A/P is H-characteristic. From the above isomorphism and corollary (10.7), it follows that $U^{-1}A/U^{-1}P$ is either zero or H-characteristic. We show that it cannot be zero. Indeed, otherwise $(U^{-1}P)(G) = (U^{-1}A)(G)$, hence $U(G)^{-1}A(G) = U(G)^{-1}P(G)$. But this contradicts the fact that $P(G)$ is disjoint from $U(G)$. In conclusion, the Green functor $U^{-1}A/U^{-1}P$ is H-characteristic. We end the proof by showing that $(U^{-1}P)(H)$ is an W_GH-prime ideal of $(U^{-1}A)(H)$. We check that the ideal $(U^{-1}P)(H)$

of $(U^{-1}A)(H)$ satisfies the hypotheses from proposition (A1.19). Let $a/r_H^G(u)$, $b/r_H^G(v) \in (U^{-1}A)(H)$ and assume that

$$\frac{a}{r_H^G(u)} \cdot c_g\Big(\frac{b}{r_H^G(v)}\Big) = \frac{a \cdot c_g(b)}{r_H^G(uv)} \in (U^{-1}P)(H), \qquad \text{for all } g \in W_GH.$$

From the above relation, it follows that, for every $g \in W_GH$, there exists $u_g \in U(G)$, such that

$$r_H^G(u_g) \cdot a \cdot c_g(b) \in P(H).$$

Let

$$u = \prod_{g \in W_GH} u_g.$$

Then

$$(r_H^G(u) \cdot a) \cdot c_g(b) \in P(H).$$

Since $P(H)$ is W_GH-prime, we conclude that $b \in P(H)$ or $r_H^G(u) \cdot a \in P(H)$. If $b \notin P(H)$, we use the fact that $r_H^G(u) \cdot a \in P(H)$, and the fact that $r_H^G(u)$ is W_GH-invariant to conclude that $a \in P(H)$ or $r_H^G(u) \in P(H)$. We now show that $r_H^G(u) \notin P(H)$. Indeed, the relation $r_H^G(u) \in P(H)$, is equivalent to $u \times 1_{A(H)} \in P(G/G \times G/H) = P(H)$. From proposition (8.15), we conclude that $u \in P(G)$, which contradicts the fact that $P(G) \cap U(G) = \emptyset$. From the above argument it follows that a or b is in $P(H)$, hence $a/r_H^G(u)$ or $b/r_H^G(v)$ is in $(U^{-1}P)(H)$.

(3) Since every maximal ideal of $U^{-1}A$ is prime, it follows that this ideal is of the form $P \cdot U^{-1}A$ for some P prime ideal of A such that $P(G) \cap U(G) = \emptyset$. Moreover, since the assignment

$$I \text{ ideal of A} \longmapsto I \cdot U^{-1}A \text{ ideal of } U^{-1}A$$

preserves containments, it follows that $P(G)$ is maximal among the prime ideals disjoint from $U(G)$ (otherwise, if $P(G) \subset q_G$ for some q_G prime ideal of $A(G)$ disjoint from $U(G)$, then $P \subset \psi_{q_G}$). Finally, since $P \subseteq \psi_{P(G)}$, from the maximality of $P \cdot U^{-1}A$, we conclude that $P = \psi_{P(G)}$.

Conversely, let p_G be a prime ideal of $A(G)$ which is maximal among the ones which are disjoint from $U(G)$. If J is an ideal of $U^{-1}A$ such that $\psi_{p_G} \cdot U^{-1}A \subset J$, we conclude that $J = I \cdot U^{-1}A$ for some ideal I of A with $\psi_{p_G} \subset I$. In particular, $p_G \subset I(G)$. If $I(G)$ is disjoint from $U(G)$, then one can use Zorn lemma to conclude that $I(G) \subseteq q_G$, for some q_G maximal among the ideals which are disjoint from $U(G)$. But it is known that every such q_G must be prime. Since $p_G \subset q_G$, this contradicts the fact that p_G was maximal among the prime ideals disjoint from $U(G)$. This shows that $I(G)$ must contain elements from $U(G)$. Hence $J = I \cdot U^{-1}A = U^{-1}A$. $\triangle$

Due to the importance of the above theorem we find it convenient to restate it as follows:

(10.9) COROLLARY.

(1) There exists a one to one correspondence between the prime ideals of $U^{-1}A$ and the prime ideals P of A such that $P(G)$ is disjoint from $U(G)$. This correspondence preserves containments and cocharacteristic subgroups.

(2) There exists a one to one correspondence between the maximal ideals of $U^{-1}A$ and the prime ideals p_G of $A(G)$ which are maximal among the ones disjoint from $U(G)$.

Let p_G be a prime ideal of $A(G)$. For $U(G) = A(G) - p_G$, the Green functor $U^{-1}A$ and the modules $U^{-1}M$ are denoted by A_{p_G} and M_{p_G}, respectively. We refer to A_{p_G} and M_{p_G} as the localizations of A and M at the prime ideal p_G of $A(G)$.

(10.10) COROLLARY.

A_{p_G} is a local Green functor. Its maximal ideal is $\psi_{p_G} \cdot A_{p_G}$.

PROOF. Immediate consequence of corollary (10.9). $\triangle$

The above corollary suggests that the localization of A at the prime ideal p_G of $A(G)$ can be thought of as the localization of A at the prime ideal ψ_{p_G} of A.

The next theorem shows that if M is a non-zero A-module then M_{m_G} is non-zero for some $m_G \in \operatorname{Max} A(G)$.

(10.11) THEOREM.

Let M be an A-module such that $M_{m_G} = 0$, for all $m_G \in \operatorname{Max} A(G)$. Then $M = 0$.

PROOF. Assume that $M \neq 0$. Let $H \in S(G)$ be such that $M(H) \neq 0$. Notice that if we regard $M(H)$ as an $A(G)$-module then $(U^{-1}M)(H)$ is the localization of $M(H)$ with respect to $U(G)$. With this convention, $(M_{m_G})(H) = M(H)_{m_G}$, for all $m_G \in \operatorname{Max} A(G)$. Since $M(H) \neq 0$, it follows that, $M(H)_{m_G} \neq 0$, for some $m_G \in \operatorname{Max} A(G)$. $\triangle$

(10.12) COROLLARY.

(1) If M_{m_G} is H-bounded for all $m_G \in \operatorname{Max} A(G)$, then M is H-bounded.

(2) Assume that for all $m_G \in \operatorname{Max} A(G)$, M_{m_G} is either zero or H-characteristic. If

M is non-zero, then M is H-characteristic.

(3) Assume that for all $m_G \in$ Max $A(G)$, A_{m_G} is X-projective. Then A is X-projective.

(4) Assume that for all $m_G \in$ Max $A(G)$, M_{m_G} satisfies X-injective induction. Then M satisfies X-injective induction.

(5) Assume that, for all $m_G \in$ Max $A(G)$, M_{m_G} is either zero or H-determined. If M is non-zero, then M is H-determined.

PROOF. *(1)* Let $X_H = \coprod_{K<H} G/K$. It is clear that a Mackey functor M is H-bounded if and only if $M_{X_H} = 0$. Since M_{m_G} is H-bounded whenever $m_G \in$ Max $A(G)$, it follows that for all $m_G \in$ Max $A(G)$,

$$0 = (M_{m_G})_{X_H} = (M_{X_H})_{m_G}.$$

From theorem (10.11), we conclude that $M_{X_H} = 0$; hence M is H-bounded.

(2) Assume that M_{m_G} is either H-characteristic or zero whenever $m_G \in$ Max $A(G)$. In particular, M_{m_G} is H-bounded, for all $m_G \in$ Max $A(G)$. Hence M is H-bounded as well. Let N be the kernel of the map

$$M \longrightarrow M_{G/H}.$$

From the exactness of the localization at m_G, it follows that N_{m_G} is the kernel of the map

$$M_{m_G} \longrightarrow (M_{G/H})_{m_G} = (M_{m_G})_{G/H}.$$

We conclude that $N_{m_G} = 0$ for all $m_G \in$ Max $A(G)$. From theorem (10.11), we conclude that $N = 0$. Now, since M is H-bounded, and the map

$$M \longrightarrow M_{G/H},$$

is injective, we conclude that M is either zero, or H-characteristic.

(3) Let M be the cokernel of the map

$$A_X \longrightarrow A.$$

Since A_{m_G} is X-projective whenever $m_G \in$ Max $A(G)$, we conclude that the map

$$(A_{m_G})_X = (A_X)_{m_G} \longrightarrow A_{m_G}$$

is surjective for all $m_G \in$ Max $A(G)$. In particular, $M_{m_G} = 0$ for all $m_G \in$ Max $A(G)$. From theorem (10.11), it follows that $M = 0$. Hence the map $A_X \longrightarrow A$ is surjective. From theorem (2.6), we conclude that A is X-projective.

(4) Follows by a similar argument applied to the sequence

$$0 \longrightarrow M \xrightarrow{d^0} M_X \xrightarrow{d^1} M_{X^2}.$$

(5) Follows from *(2)* and *(4)*. $\triangle$

We show that the primordial subgroups of an A-module M can be determined from the primordial subgroups of the modules M_{m_G} for $m_G \in$ Max $A(G)$.

(10.13) PROPOSITION.

(1) If $U(G)$ is a multiplicative subset of $A(G)$ then $\mathcal{P}(U^{-1}M) \subseteq \mathcal{P}(M)$.

(2)

$$\mathcal{P}(M) = \bigcup_{m_G \in \mathrm{Max} A(G)} \mathcal{P}(M_{m_G}).$$

PROOF. *(1)* Assume $H \notin \mathcal{P}(A)$. According to proposition (4.1.17), the map

$$M_{X_H} \xrightarrow{\pi_H^*} M_{G/H}$$

is onto, where π_H^* is the canonical map given by formula (4.12). Since the functor U^{-1} is exact, it follows that the map

$$U^{-1}(\pi_H^*) : U^{-1}(M_{X_H}) \longrightarrow U^{-1}(M_{G/H})$$

is onto. By proposition (10.6), the above map can be regarded as the map π_H^* for the functor $U^{-1}M$. From proposition (4.1.17) it follows that $H \notin \mathcal{P}(U^{-1}(M))$.

(2) From *(1)* it follows that

$$\bigcup_{m_G \in \mathrm{Max} A(G)} \mathcal{P}(M_{m_G}) \subseteq \mathcal{P}(M).$$

We now prove the reverse containment. Assume

$$H \notin \bigcup_{m_G \in \mathrm{Max} A(G)} \mathcal{P}(M_{m_G}).$$

Let N be the cokernel of the map π_H^*. Since H is not primordial for any M_{m_G}, it follows, from proposition (4.1.17), that $N_{m_G} = 0$ for all $m_G \in$ Max $A(G)$. Hence $N = 0$, which shows that π_H^* is surjective. From lemma (4.1.17) we conclude that $H \notin \mathcal{P}(M)$. $\triangle$

Suppose that A is an H-characteristic Green functor. Assume that $0 \notin U(G)$. Then $U^{-1}A$ is also H-characteristic. Indeed, otherwise, from corollary (10.7), $U^{-1}A$ is zero. In particular, $(U^{-1}A)(G) = U(G)^{-1}A(G) = 0$, which contradicts the fact that $0 \notin U(G)$.

When A is a domain, we have the following result.

(10.14) PROPOSITION.

If A is a domain and $0 \notin U(G)$, then

$$j_A : A \longrightarrow U^{-1}A$$

is injective.

PROOF. Let H be the characteristic subgroup of A, and let I be the kernel of j_A. If I is non-zero, it follows that I is H-characteristic as well. In particular, $I(H) \neq 0$. Notice that $I(H)$ is the kernel of the map

$$A(H) \longrightarrow (U^{-1}A)(H) = U(H)^{-1}A(H).$$

Suppose that x is a non-zero element in $I(H)$. Then there exists $u \in U(G)$ such that $r^G_H(u) \cdot x = 0$. Since $r^G_H(u)$ is W_GH-invariant and the ideal 0 is W_GH-prime, we use proposition (A1.20)*(2)* to conclude that $x = 0$ or $r^G_H(u) = 0$. Since $x \neq 0$, it follows that $r^G_H(u) = 0$. This is equivalent to $u \times 1_{A(H)} = 0 \in A(G/G \times G/H)$. By proposition (8.15) it follows that $u = 0$. This contradicts the fact that $0 \notin U(G)$. $\triangle$

(10.15) DEFINITION.

Let A be domain, and let $U(G) = A(G) - \{0\} = A(G)^*$. The Green functor $U^{-1}A$ is called the *fraction functor* of A and is denoted $F(A)$.

Notice that A can be regarded as a subfunctor of $F(A)$ via the injection j_A. Moreover, for every $m_G \in \operatorname{Max} A(G)$, the localized Green functors A_{m_G} can be regarded as intermediary subfunctors $A \hookrightarrow A_{m_G} \hookrightarrow F(A)$. With these conventions, we have the following result.

(10.16) PROPOSITION.

If A is a domain, then

$$A = \bigcap_{m_G \in \operatorname{Max} A(G)} A_{m_G}.$$

PROOF. Let X be a finite G-set and let $x \in (F(A))(X)$. We set

$$I(Y) = \{a \in A(Y) \mid a \times x \in A(Y \times X)\}, \qquad \text{for all } Y \in G\text{-}Set.$$

It is clear that I is an ideal of A. For $p_G \in \mathrm{Spec}(A(G))$, the condition $x \in A_{p_G}$ is equivalent to $I \not\subseteq \psi_{p_G}$. In particular, if

$$x \in \bigcap_{m_G \in \mathrm{Max}\ A(G)} A_{m_G}$$

then $I \not\subseteq \psi_{m_G}$ for any $m_G \in \mathrm{Max}\ A(G)$. This implies that I is not contained in any maximal ideal of A; hence $I = A$. This shows that $1_{A(G)} \in I(G)$; hence $x = 1_{A(G)} \times x \in A(G/G \times X) = A(X)$. The above argument shows that

$$\bigcap_{m_G \in \mathrm{Max} A(G)} A_{m_G} \subseteq A.$$

The reverse containment is obvious. Δ

Let A be a domain and assume that A' is a field containing A. Then, by proposition (10.3) and (10.15), A' contains $F(A)$. Unfortunately, $F(A)$ need not be a field as is shown by the following example.

(10.17) EXAMPLE. Let p be a prime number, and let $G = \mathbf{Z}_p$. Let $A(G) = A(1) = \mathbf{Z}_p$ be the Green functor with r_1^G the identity map of $\mathbf{Z}_p$ and $t_1^G = 0$. It is clear that A is a domain which is not a field and $A = FA$.

We conclude this chapter with a necessary and sufficient condition for $F(A)$ to be a field.

(10.18) PROPOSITION.

$F(A)$ is a field if and only if $0 = \psi_0$ in A.

PROOF. Immediate consequence of corollary (10.10). Δ

(10.19) THEOREM.

Let A be an H-characteristic domain. Then $F(A)$ is a field if and only if there exists $a \in A(H)$ such that $\sum_{g \in W_G H} c_g(a) \neq 0$.

PROOF. Notice that since

$$\sum_{g \in W_G H} c_g(a) = r_H^G(t_H^G(a)) \qquad \text{for all } a \in A(H),$$

the condition from the hypothesis is equivalent to the following:

There exists $a \in A(H)$ and $u \in A(G)^*$ such that $\sum_{g \in W_G H} c_g(a) = r_H^G(u)$.

We prove the result under the above form. Assume that $F(A)$ is a field. By theorem (9.1.8), the algebra $(F(A))(H) = \left(A(G)^{*}\right)^{-1}A(H)$ is projective. Hence there exists an element $a/r_H^G(u) \in \left(A(G)^{*}\right)^{-1}A(H)$ such that

$$1_{(F(A))(H)} = \sum_{g\in W_G H} c_g\Big(\frac{a}{r_H^G(u)}\Big) = \frac{\sum_{g\in W_G H} c_g(a)}{r_H^G(u)}.$$

It follows that for some $v \in A(G)$,

$$\big(r_H^G(u) - \sum_{g\in W_G H} c_g(a)\big)\cdot r_H^G(v) = 0.$$

The above relation can be rewritten as

$$\sum_{g\in W_G H} c_g\big(a\cdot r_H^G(v)\big) = r_H^G(u\cdot v).$$

Conversely, assume that there exists $a \in A(H)$ and $u \in A(G)^{*}$ such that

$$\sum_{g\in W_G H} c_g(a) = r_H^G(u). \tag{10.1}$$

We check that $F(A)$ satisfies the hypotheses of theorem (9.1.8). If we regard equation (10.1) in $(F(A))(H)$, then

$$1_{(F(A))(H)} = \sum_{g\in W_G H} c_g\Big(\frac{a}{r_H^G(u)}\Big).$$

Hence the algebra $(F(A))(H)$ is projective. We now show that $(F(A))(H)$ is a W_GH-field. By proposition (A1.21), it is enough to check that every non-zero W_GH-invariant element of $(F(A))(H)$ is a unit in $(F(A))(H)$. Let $x = b/r_H^G(v)$ be a non-zero W_GH-invariant element of $(F(A))(H)$. Then b is a non-zero element of $A(H)^{W_GH}$. It is enough to show that b is invertible in $(F(A))(H)$. If we multiply (10.1) by b we obtain

$$b\cdot r_H^G(u) = b\cdot\Big(\sum_{g\in W_G H} c_g(a)\Big) = \sum_{g\in W_G H} c_g(ba) = r_H^G\big(t_H^G(ba)\big). \tag{10.2}$$

Since $A(H)^{W_GH} - \{0\}$ does not contain W_GH-invariant zero divisors (by proposition (A1.20)*(2)*), it follows that $t_H^G(ba) \neq 0$; hence $t_H^G(ba) \in A(G)^{*}$. From equation (10.2), it follows that b is invertible in $(F(A))(H)$. $\triangle$

Let A be an H-characteristic domain. By theorem (9.1.2), $0 = A_{(H,\, \mathrm{eq}(p_H))}$ for some prime ideal p_H of $A(H)$. Let

$$W_G(H,p_H) = \{g \in W_G H \mid {}^g p_H = p_H\}$$

and let $K(H,p_H)$ be the kernel of the action of $W_G(H,p_H)$ on the domain $A(H)/p_H$. Theorem (10.19) is equivalent to the following:

(10.20) THEOREM.

Let A be an H-characteristic domain. Suppose $0 = A_{(H,\, \mathrm{eq}(p_H))}$ for some prime ideal p_H of $A(H)$. Let p be the characteristic of the domain $A(H)/p_H$. Then $F(A)$ is a field if and only if p does not divide the order of $K(H,p_H)$.

PROOF. Follows immediately from theorem (10.19) and proposition (A1.25). △

11. Krull Dimension.

Throughout this chapter, A is a commutative Green functor and P, Q are prime ideals of A. We investigate the containments among prime ideals of A. In particular, if $P \subsetneq Q$ are prime ideals of A we indicate a procedure for finding prime ideals $\widetilde{P}$ such that $P \subsetneq \widetilde{P} \subsetneq Q$. Let S be a commutative ring. In commutative algebra, the proper way to investigate the prime ideals between two prime ideals $p \subsetneq q$ of S is to factor out p and localize with respect to q. The prime ideals in the new ring obtained in this way are in a one-to-one correspondence with the intermediary prime ideals of S lying between p and q. The advantage of this method is that it reduces a problem with constraints (namely finding intermediary prime ideals between p and q) to computing the arbitrary prime ideals of a local domain. Assume now that $P \subsetneq Q$ are two prime ideals of A. It follows, from the results from chapter 10, that one can use the same procedure for detecting intermediary prime ideals of A provided that $Q = \psi_{Q(G)}$. Unfortunately, the method fails if $Q \subset \psi_{Q(G)}$. Now we outline a ***restriction procedure*** which reduces the case $Q \subset \psi_{Q(G)}$ to a case for which Q is maximal in its fibre.

Assume that $H \in S(G)$ and M is an A-module. Let N' be a sub-$A\downarrow^G_H$-submodule of $M\downarrow^G_H$. We show that N' leads naturally to a submodule of M denoted $N = (\downarrow^G_H)^{-1}(N')$. We refer to N as ***sitting over*** N'. If N' is cocharacteristic then N is cocharacteristic as well. Conversely, if $K \leq H$ and N is a K-cocharacteristic submodule of M then, in general, $N\downarrow^G_H$ is not K-cocharacteristic. However, we show that $N\downarrow^G_H$ is an intersection of finitely many submodules $\{N'_i\}_{i=1}^n$ of $M\downarrow^G_H$ such that N sits over each N'_i and the submodule N'_i is ${}^{g_i}K$-cocharacteristic for some $g_i \in G$. Now if P is a K-cocharacteristic prime ideal of A then we show that $P\downarrow^G_H$ is a radical ideal of $A\downarrow^G_H$ which is an intersection of finitely many prime prime ideals $\{P'_i\}_{i=1}^m$ such that P sits over each P'_i and the ideal P'_i is ${}^{g_i}K$-cocharacteristic for some $g_i \in G$. We then show that if $P \subseteq Q$ are prime ideals of A and H is a subgroup containing a cocharacteristic subgroup of Q, then there exists prime ideals $P' \subseteq Q'$ of $A\downarrow^G_H$ such that P and Q sit over P' and Q' respectively. Assume now that Q is H-cocharacteristic. Then the prime ideal Q' of $A\downarrow^G_H$ constructed above is maximal in its fibre. In particular, the intermediary prime ideals between P' and Q' can be studied by factoring P' and localizing with respect to $Q'(H)$. Finally, every intermediary prime ideal between P' and Q' will lead, via the correspondence $(\downarrow_H^G)^{-1}$, to intermediary primes of A between P and Q. The main result that we obtain using the above technique is theorem (11.17). This result shows that containments among prime ideals of A resemble the containments among the prime ideals of the Burnside ring Green functor (see [Le1]). We also give very precise results when A is a Green functor satisfying a certain integrality assumption.

We begin with some general considerations concerning the Krull dimension of A.

Let $\mathcal{S}$ be a subset of Spec(A). The supremum of the lengths r taken over all strictly increasing chains

$$P_0 \subset P_1 \subset \ldots \subset P_r, \qquad P_i \in \mathcal{S}, \text{ for } 0 \leq i \leq r$$

is called the *Krull dimension* of $\mathcal{S}$, or simply the dimension of $\mathcal{S}$. It is denoted by $\dim(\mathcal{S})$.

When $P \in \mathcal{S}$, the dimension of the set $\{Q \in \mathcal{S} \mid Q \subseteq P\}$ is denoted with $\mathrm{ht}_{\mathcal{S}}(P)$ and called the *height of P in $\mathcal{S}$*.

Similarly, the dimension of the set $\{Q \in \mathcal{S} \mid P \subseteq Q\}$ is denoted with $\mathrm{coht}_{\mathcal{S}}(P)$ and called the *coheight of P in $\mathcal{S}$*.

We introduce the following definition.

(11.1) DEFINITION. Assume that $P \subset Q$ are two prime ideals.

(i) the containment $P \subset Q$ is called *of the first kind* if the domains A/P and A/Q have the same characteristic subgroup.

(ii) the containment $P \subset Q$ is called *of the second kind* if it is not of the first kind. Moreover, the containment is called *purely of the second kind* if, whenever $P \subset \widetilde{P} \subseteq Q$, the containment $P \subset \widetilde{P}$ is of the second kind.

Notice that, by corollary (9.1.7), if $P \subset Q$ is a containment of the second kind and $Q \subseteq \widetilde{Q}$, then the containment $P \subset \widetilde{Q}$ is of the second kind as well.

For $\mathcal{S} \subseteq$ Spec(A), a chain

$$P_0 \subset P_1 \subset \ldots \subset P_r$$

of prime ideals in $\mathcal{S}$ is called of the *first*, respectively *second kind*, if all the containments $P_i \subset P_{i+1}$, for $0 \leq i \leq r-1$, are of the first, respectively second kind. The first, respectively second Krull dimension of $\mathcal{S}$ is the supremum of the lengths r of chains of first, respectively second kind in $\mathcal{S}$. They are denoted $\dim_1(\mathcal{S})$ and respectively $\dim_2(\mathcal{S})$. For $P \in \mathcal{S}$, one can similarly define $\mathrm{ht}_{i,\mathcal{S}}(P)$ and $\mathrm{coht}_{i,\mathcal{S}}(P)$ for $i = 1, 2$. Notice that $\dim_2(\mathcal{S})$is finite, and

$$\dim(\mathcal{S}) \leq \dim_1(\mathcal{S}) \cdot \dim_2(\mathcal{S}). \qquad (11.1)$$

Moreover, notice that $\dim_1(S)$ is closely related to the classical notion of Krull dimension. Indeed, assume that $H \in \mathcal{P}(A)$, and let

$$P_0 \subset P_1 \subset \ldots \subset P_r, \qquad P_i \in S, \text{ for all } 0 \leq i \leq r \tag{11.2}$$

be a chain of prime ideals in S such that H is the common characteristic subgroup of all the domains A/P_i, for $1 \leq i \leq r$. From corollary (9.2),

$$P_i = A_{(H,\ \mathrm{eq}(\overline{p_{i,H}}))}$$

for some $\overline{p_{i,H}} \in \operatorname{Spec} \overline{A(H)}$. Since the operator $A_{(H,\ -)}$ reflects containments, we conclude that

$$\mathrm{eq}(\overline{p_{i,H}}) \subset \mathrm{eq}(\overline{p_{i+1,H}}), \qquad \text{for all } 1 \leq i \leq r-1.$$

If we fix i, the above containment is obviously equivalent to

$$\overline{p_{i,H}} \subset {}^{g}\overline{p_{i+1,H}}, \qquad \text{for some } g \in W_G H.$$

However, since $\mathrm{eq}({}^{g}\overline{p}) = \mathrm{eq}(\overline{p})$ for all $g \in W_G H$ and $\overline{p} \in \operatorname{Spec} \overline{A(H)}$, we can assume that

$$\overline{p_{i,H}} \subset \overline{p_{i+1,H}}, \qquad \text{for all } 1 \leq i \leq r-1.$$

Assume now that one wants to compute the Krull dimension of the Green functor A. From the above argument, one concludes immediately that

$$\dim_1(A) \leq \max_{H \in \mathcal{P}(A)} \left(\dim \overline{A(H)}\right). \tag{11.3}$$

Moreover, from corollary (9.1.7), it follows that, if

$$P_0 \subset P_1 \subset \ldots \subset P_r$$

is a chain of prime ideals of the second kind in A, then there exists a prime number $p \mid |G|$, which is not invertible in $A(G)$, such that p is the common characteristic of the rings $A(G)/P_i(G)$ for $1 \leq i \leq r$. In this case, the ring $A(G)/P_0(G)$ has characteristic zero or p. Moreover, by corollary (9.1.7) again, it follows that, if H_i is the characteristic subgroup of A/P_i, then

$$[H] = [H_0] < [H_1] < \ldots < [H_r] \leq [H^p].$$

In particular $r \leq \log_p\left(|H^p/H|\right)$. From inequalities (11.1) and (11.3), we conclude that, if Π

$$\dim(A) \leq \max\Big(\log_p\left(|H^p/H|\right) \mid H \in \mathcal{P}(A),\ p \in \Pi\Big) \cdot \max_{H \in \mathcal{P}(A)} \left(\dim \overline{A(H)}\right).$$

In the remainder of this chapter we derive more precise bounds for the Krull dimension of a Green functor A.

We begin with the description of the *restriction procedure.*

(11.2) PROPOSITION.

Let $H \in S(G)$ and $K \leq H$.

(1) If $M' \in Mack_R(H)$ is K-characteristic, then $M'\uparrow_H^G$ is K-characteristic.

(2) If A' is an H-Green functor which is a K-characteristic domain, then $A'\uparrow_H^G$ is also a K-characteristic domain.

PROOF. *(1)* From formula (2.1), it follows immediately that $(M'\uparrow_H^G)(L) = 0$, if $L < K$. Moreover, $(M'\uparrow_H^G)(K)$ contains a copy of $M'(K)$ as a direct summand. In particular, $M'\uparrow_H^G \in K\text{-}Mack_R(G)$, and $(M'\uparrow_H^G)(K) \neq 0$. Since the map

$$M' \longrightarrow M'_{H/K}$$

is injective, and since the H-set $(G/K)\downarrow_H^G$ contains orbits of the form H/K, it follows that the map

$$M' \longrightarrow M'_{(G/K)\downarrow_H^G}$$

is injective. It follows, by applying the functor $\uparrow_H^G$, that the map

$$M'\uparrow_H^G \longrightarrow (M'_{(G/K)\downarrow_H^G})\uparrow_H^G \tag{11.4}$$

is injective as well. Now, if X is a finite G-set, then

$$(M'_{(G/K)\downarrow_H^G})\uparrow_H^G(X) = M'_{(G/K)\downarrow_H^G}(X\downarrow_H^G) = M'((G/K)\downarrow_H^G \times X\downarrow_H^G)$$

$$= M'((G/K \times X)\downarrow_H^G) = (M'\uparrow_H^G)(G/K \times X) = (M'\uparrow_H^G)_{G/K}(X).$$

In particular, the functor from the right side of formula (11.4) can be identified with

$$(M'\uparrow_H^G)_{G/K}.$$

It is clear that, with this identification, map (11.4) is the map $\theta^{G/K}$ for $M'\uparrow_H^G$.

(2) From *(1)* we know that $A'\uparrow_H^G$ is K-characteristic. We show that $(A'\uparrow_H^G)(K)$ is a W_GK-domain. We apply formula (2.1) to conclude that

$$(A'\uparrow_H^G)(K) = \bigoplus_{g\in[H\backslash G/K]} A'(H \cap {}^gK).$$

Since A' is K-bounded, it follows that $A'(H \cap {}^gK) = 0$ unless ${}^gK = {}^hK$ for some $h \in H$. In conclusion

$$(A' \uparrow_H^G)(K) = \bigoplus_{\{g \in [H \backslash G/K] \mid {}^gK = {}^hK \text{ for some } h \in H\}} A'(K).$$

It can be easily seen that the above indexing set can be regarded as $W_GK \backslash W_HK$. With this convention, we obtain that

$$(A' \uparrow_H^G)(K) = \bigoplus_{g \in W_GK \backslash W_HK} A(K).$$

If one thinks of $A'(K)$ as an R-linear representation of W_HK, then $(A' \uparrow_H^G)(K)$ can be identified with the induced representation $\mathrm{Ind}_{W_HK}^{W_GK} A'(K)$. Since $A'(K)$ is an W_HK-domain, it follows immediately that $(A' \uparrow_H^G)(K)$ is an W_GK-domain. $\triangle$

(11.3) DEFINITION (The $(\downarrow_H^G)^{-1}$ Operator). Let $H \leq G$ and let M be an A-module. Assume that N' is a sub-$A \downarrow_H^G$-module of $M \downarrow_H^G$. Let π' be the canonical projection

$$M \downarrow_H^G \xrightarrow{\pi'} \frac{M \downarrow_H^G}{N'}.$$

This map is a map of $A \downarrow_H^G$-modules. If we apply the functor $\uparrow_H^G$ we obtain a map

$$(M \downarrow_H^G) \uparrow_H^G \xrightarrow{\pi' \uparrow_H^G} \left(\frac{M \downarrow_H^G}{N'} \right) \Bigg\uparrow_H^G . \tag{11.5}$$

Notice that $(M \downarrow_H^G) \uparrow_H^G = M_{G/H}$ and $\pi' \uparrow_H^G$ is a map of $A_{G/H}$-modules. If we regard the modules from formula (11.5) as A-modules, then $\pi' \uparrow_H^G$ is a map of A-modules. The submodule $N = (\downarrow_H^G)^{-1}$ of M is the kernel of the composition

$$M \xrightarrow{\theta^{G/H}} M_{G/H} \xrightarrow{\pi \uparrow_H^G} \left(\frac{M \downarrow_H^G}{N'} \right) \Bigg\uparrow_H^G . \tag{11.6}$$

We refer to $(\downarrow_H^G)^{-1}(N')$ as *sitting over* N'. Hence, $(\downarrow_H^G)^{-1}$ is an operator from the lattice of sub-$A \downarrow_H^G$-modules of $M \downarrow_H^G$ to the lattice of submodules of M.

The main properties of the operator $(\downarrow_H^G)^{-1}$ are summarized in the following three propositions.

(11.4) PROPOSITION.

Let M be an A-module and let N' be a submodule of $M \downarrow_H^G$. Let $N = (\downarrow_H^G)^{-1}(N')$. Then $N \downarrow_H^G \subseteq N'$.

PROOF. From chapter 2 we now that $\uparrow_H^G$ is both left and right adjoint to $\downarrow_H^G$. If M' is an H-Mackey functor over R, let $\omega : (M' \uparrow_H^G)\downarrow_H^G \longrightarrow M'$ be the counit of this adjunction.

Now let M be an A-module and let N' be a submodule of $M\downarrow_H^G$. If we apply the functor $\downarrow_H^G$ to the canonical map

$$\frac{M}{N} = \frac{M}{(\downarrow_H^G)^{-1}(N')} \longrightarrow \left(\frac{M\downarrow_H^G}{N'}\right)\Big\uparrow_H^G$$

we obtain a map of $A\downarrow_H^G$-modules

$$\left(\frac{M}{N}\right)\Big\downarrow_H^G = \frac{M\downarrow_H^G}{N\downarrow_H^G} \quad\longrightarrow\quad \left(\frac{M\downarrow_H^G}{N'}\right)\Big\uparrow_H^G\Big\downarrow_H^G .$$

It is easy to see that this map has a factorisation

$$\begin{tikzcd} \frac{M\downarrow_H^G}{N\downarrow_H^G} \arrow[r] & \left(\frac{M\downarrow_H^G}{N'}\right)\Big\uparrow_H^G\Big\downarrow_H^G \\ & \frac{M\downarrow_H^G}{N'} \arrow[ul, "\psi"] \arrow[u, "\omega"'] \end{tikzcd}$$

where the map ψ is a canonical projection. Moreover, it is easy to see that in the above diagram all maps are maps of $A\downarrow_H^G$-modules. Since the map ψ is a projection, it follows that $N\downarrow_H^G \subseteq N'$. $\quad\triangle$

(11.5) PROPOSITION.

Let $K \leq H \leq G$ and let M be an A-module.

(1) Assume that N' is a K-cocharacteristic submodule of $M\downarrow_H^G$. Then $N = (\downarrow_H^G)^{-1}(N')$ is a K-cocharacteristic submodule of M. More precisely, if

$$N' = (M\downarrow_H^G)_{(K,\ \overline{N'_K})}$$

for some W_HK-invariant submodule $\overline{N'_K}$ of $\overline{M(K)}$, then

$$N = M_{(K,\ \mathrm{eq}(\overline{N'_K}))}. \tag{11.7}$$

(2) The operator $(\downarrow_H^G)^{-1}$ preserves containments.

(3) If N' and N'_1 are two distict K-cocharacteristic submodules of $M\downarrow_H^G$ such that $(\downarrow_H^G)^{-1}(N') = (\downarrow_H^G)^{-1}(N'_1)$, then there is no containment relation between N' and N'_1.

(4) Let N be a K-cocharacteristic submodule of M. Then there exists a finite family $\{N_i'\}_{i=1}^n$ of cocharacteristic submodules of $M\downarrow_H^G$ such that

$$N\downarrow_H^G = \bigcap_{i=1}^n N_i'.$$

Moreover, N_i' can be chosen such that $(\downarrow_H^G)^{-1}(N_i') = N$.

More precisely, if $N = M_{(K,\ \overline{N_K})}$, then

$$N\downarrow_H^G = \bigcap_{g\in D(H,K)} (M\downarrow_H^G)_{({}^gK,\ {}^g\overline{N_K})}. \tag{11.8}$$

PROOF. *(1)* The map

$$\frac{M}{N} \longrightarrow \left(\frac{M\downarrow_H^G}{N'}\right)\Big\uparrow_H^G$$

is injective. Since $M\downarrow_H^G/N'$ is K-characteristic, it follows, by proposition (11.2) *(1)*, that $\left(M\downarrow_H^G/N'\right)\Big\uparrow_H^G$ is K-characteristic. Since M/N can be regarded as a submodule of $\left(M\downarrow_H^G/N'\right)\Big\uparrow_H^G$ it follows, by proposition (6.1.3) *(1)*, that M/N is K-characteristic as well. Formula (11.7) is straightforward.

(2) Let $N' \subseteq N_1'$. Then *(2)* follows from the commutativity of the diagram

$$\begin{array}{ccc}
M & \longrightarrow & \left(\dfrac{M\downarrow_H^G}{N_1'}\right)\Big\uparrow_H^G \\
\Big\uparrow 1_M & & \Big\uparrow \pi_1'\uparrow_H^G \\
M & \longrightarrow & \left(\dfrac{M\downarrow_H^G}{N'}\right)\Big\uparrow_H^G
\end{array}$$

where $\pi_1' : M\downarrow_H^G/N' \longrightarrow M\downarrow_H^G/N_1'$ is the canonical projection.

(3) Immediate consequence of *(2)* and formula (11.7).

(4) If we apply the functor $\downarrow_H^G$ to the injection

$$\frac{M}{N} \longrightarrow \left(\frac{M}{N}\right)_{G/K}$$

we obtain an injection

$$\frac{M\downarrow_H^G}{N\downarrow_H^G} \longrightarrow \left(\left(\frac{M}{N}\right)_{(G/K)}\right)\Big\downarrow_H^G = \left(\frac{M\downarrow_H^G}{N\downarrow_H^G}\right)_{(G/K)\downarrow_H^G}.$$

Since

$$(G/K)\downarrow_H^G = \coprod_{g\in[H\backslash G/K]} H/H\cap {}^gK,$$

it follows, from the fact that M/N is K-bounded, that the above map is in fact the map

$$\frac{M\downarrow_H^G}{N\downarrow_H^G} \longrightarrow \bigoplus_{g\in D(H,K)} \left(\frac{M\downarrow_H^G}{N\downarrow_H^G}\right)_{H/{}^gK}. \tag{11.9}$$

For $g \in D(H,K)$, let N'_g be the submodule of $M\downarrow_H^G$ containing $N\downarrow_H^G$ such that $N'_g/(N\downarrow_H^G)$ is the kernel of the map

$$\frac{M\downarrow_H^G}{N\downarrow_H^G} \longrightarrow \left(\frac{M\downarrow_H^G}{N\downarrow_H^G}\right)_{H/{}^gK}.$$

Then N'_g is ${}^gK$-cocharacteristic and $(\downarrow_H^G)^{-1}(N'_g) = N$. Moreover, from formula (11.9), it follows that

$$N\downarrow_H^G = \bigcap_{g\in D(H,K)} N'_g.$$

Formula (11.8) is straightforward. Δ

(11.6) PROPOSITION.

Let $K \leq H \leq G$ and A be a commutative Green functor.

(1) If P' is a K-cocharacteristic prime ideal of $A\downarrow_H^G$ then $P = (\downarrow_H^G)^{-1}(P')$ is a K-cocharacteristic prime ideal of A. More precisely, if

$$P' = (A\downarrow_H^G)_{(K,\ \overline{P_K})}$$

for some W_HK-prime ideal $\overline{P_K}$ of $\overline{A(K)}$, then

$$P = A_{(K,\ \mathrm{eq}(\overline{P_K}))}. \tag{11.10}$$

(2) Assume that P' and P'_1 are two K-cocharacteristic prime ideals of $A\downarrow_H^G$ such that $(\downarrow_H^G)^{-1}(P') = (\downarrow_H^G)^{-1}(P'_1)$. Then there is no containment relation between P' and P'_1.

(3) Let P be a K-cocharacteristic prime ideal of A. Then there exists a finite family $\{P'_i\}_{i=1}^m$ of prime ideals of $A\downarrow_H^G$ such that

$$P\downarrow_H^G = \bigcap_{i=1}^m P'_i.$$

Moreover, P'_i can be chosen such that $(\downarrow^G_H)^{-1}(P'_i) = P$. More precisely, if $P = A_{(K,\ \mathrm{eq}(\overline{p_K}))}$, for some prime ideal $\overline{p_K}$ of $\overline{A(K)}$, then

$$P\downarrow^G_H = \bigcap_{g\in D(H,K)} \ \bigcap_{w\in W_G{}^gK\backslash W_H{}^gK} (A\downarrow^G_H)_{\left({}^gK,\ \bigcap_{t\in W_H{}^gK} {}^t({}^w\overline{p_K})\right)}. \tag{11.11}$$

(4) If P is a prime ideal of A and P' is a prime ideal of $A\downarrow^G_H$ such that P sits over P', then P' must be one of the prime ideals appearing in the right hand side of formula (11.11).

PROOF. Similar to the proof of (11.5) *(1)*.

(2) Obvious consequence of formula (11.10).

(3) Let $P = A_{(K,\ \mathrm{eq}(\overline{p_K}))}$. By formula (11.8)

$$P\downarrow^G_H = \bigcap_{g\in D(H,K)} (A\downarrow^G_H)_{({}^gK,\ {}^g\mathrm{eq}(\overline{p_K}))}.$$

In order to derive formula (11.11) it is enough to show that

$$(A\downarrow^G_H)_{({}^gK,\ {}^g\mathrm{eq}(\overline{p_K}))} = \bigcap_{w\in W_G{}^gK\backslash W_H{}^gK} (A\downarrow^G_H)_{\left({}^gK,\ \bigcap_{t\in W_H{}^gK} {}^t({}^w\overline{p_K})\right)} \quad \text{for } g\in D(H,K). \tag{11.12}$$

We prove (11.12) when $g = HeK$. In this case, (11.12) becomes

$$(A\downarrow^G_H)_{(K,\ \mathrm{eq}(\overline{p_K}))} = \bigcap_{w\in W_GK\backslash W_GH} (A\downarrow^G_H)_{\left(K,\ \bigcap_{t\in W_HK} {}^t({}^w\overline{p_K})\right)}.$$

But this last formula follows from the fact that

$$\mathrm{eq}(\overline{p_K}) = \bigcap_{g\in W_GK} {}^g\overline{p_K} = \bigcap_{g\in W_GK\backslash W_HK}\ \bigcap_{t\in W_HK} {}^t({}^w\overline{p_K}),$$

and from the fact that the operator $(A\downarrow^G_H)_{(K,\ -)}$ commutes with intersections.

(4). Let P' be such that $(\downarrow^G_H)^{-1}(P') = P$. Then, from proposition (11.4), it follows that

$$P' \supseteq P\downarrow^G_H = \bigcap_{i=1}^m P'_i.$$

Since P' is a prime ideal it follows that for some i $P'_i \subseteq P'$. Since P sits over both P' and P'_i, it follows, by *(2)*, that $P' = P'_i$. $\triangle$

(11.7) THEOREM (The Going-Down Restriction Theorem).

Let $P \subset Q$ be two prime ideals of A. Assume that Q is K-cocharacteristic and that $K \leq H$.

(1) Let $Q' \in$ Spec $(A\downarrow_H^G)$ be such that Q sits over Q'. Then there exists $P' \in$ Spec $(A\downarrow_H^G)$ such that $P' \subset Q'$ and P sits over P'. If the containment $P \subset Q$ is of the first (second) kind, then $P' \subset Q'$ is also of the first (second) kind. If the containment $P \subset Q$ is purely of the second kind, then the containment $P' \subset Q'$ is purely of the second kind.

(2) Again let $Q' \in$ Spec $(A\downarrow_H^G)$ be such that Q sits over Q'. Assume that $\{P_i'\}_{i=1}^n$ are all the prime ideals of $A\downarrow_H^G$ such that P sits over P_i' and $P_i' \subset Q'$. If all the containments $P_i' \subset Q'$ are purely of the second kind, then the containment $P \subset Q$ is also purely of the second kind.

PROOF. *(1).* Assume that $\{P_i'\}_{i=1}^m$ are all the prime ideals of $A\downarrow_H^G$ such that P sits over P_i'. From proposition (11.4), it follows that

$$Q' \supseteq Q\downarrow_H^G \supseteq P\downarrow_H^G = \bigcap_{i=1}^m P_i'$$

Since Q' is prime, it follows that for some $1 \leq i \leq m$, $Q' \supseteq P_i'$. It is clear that in fact $Q' \supset P_i'$ (otherwise $Q = (\downarrow_H^G)^{-1}(Q') = (\downarrow_H^G)^{-1}(P_i') = P$).

(2) Assume that $P \subsetneq \widetilde{P} \subset Q$ is a chain of prime ideals such that $P \subsetneq \widetilde{P}$ is of the first kind. Let Q' be a prime ideal of $A\downarrow_H^G$ such that Q sits over Q'. We use *(1)* to construct $\widetilde{P}'$ such that $\widetilde{P}' \subset Q'$ and $\widetilde{P}$ sits over $\widetilde{P}'$. We apply *(1)* again to construct $P' \in$ Spec $(A\downarrow_H^G)$ such that P sits over P' and $P' \subsetneq \widetilde{P}'$. In the chain $P' \subsetneq \widetilde{P}' \subset Q'$, the containment $P' \subsetneq \widetilde{P}'$ is obviously of the first kind. Since the containment $P' \subset Q'$ is purely of the second kind, it follows that $P' = \widetilde{P}'$, hence $P = (\downarrow_H^G)^{-1}(P') = (\downarrow_H^G)^{-1}(\widetilde{P}') = \widetilde{P}$. In conclusion, $P \subset Q$ is purely of the second kind. $\triangle$

We introduce the folowing definition.

(11.8) DEFINITION. Let $A \xrightarrow{f} A'$ be a unitary morphism of commutative Green functors. The functor A' is *integral* over A if $A'(H)$ is integral over $A(H)$ via the map $f(H)$ for all $H \in S(G)$. A commutative Green functor A is said to be *integral* if it is integral over its twin.

The following proposition summarizes the main properties of integral Green functors.

(11.9) PROPOSITION.

The following conditions are equivalent:

(1) A is integral.

(2) $\overline{A(H)}$ is integral over $A(G)$ via the restriction map $br_H \cdot r^G_H$ for all $H \in \mathcal{P}(A)$.

(2) $A(H)$ is integral over $A(G)$ via the restriction map r^G_H for all $H \in S(G)$.

PROOF. (1) $\Rightarrow$ (2) From formula (4.15) it follows easily that

$$(T(A))(G) = \prod_{H \in [G \backslash \mathcal{P}(A)]} \overline{A(H)}^{W_G H} \tag{11.13}$$

is integral over $A(G)$ via the map $\beta(G)$. If $H \in \mathcal{P}(A)$ it follows, from formula (11.13), that $\overline{A(H)}^{W_G H}$ is a subring of $(T(A))(G)$. In particular $\overline{A(H)}^{W_G H}$ is integral over $A(G)$. Now (2) follows from the fact that $\overline{A(H)}$ is integral over $\overline{A(H)}^{W_G H}$.

(2) $\Rightarrow$ (3). Let $H \in S(G)$, and let $x \in A(H)$. For each $K \in \mathcal{P}(A) \cap S(H)$, let $F_K(X) \in \overline{A(K)}[X]$,

$$\mathcal{F}_K(X) = X^{n_K} + \sum_{i=1}^{n_K} br_K \cdot r^G_K(a_{K,i}) \cdot X^{n_K - i}, \qquad a_{K,i} \in A(G) \text{ for } 1 \leq i \leq n_K$$

be a monic polynomial such that $br_K \cdot r^H_K(x)$ is a root of $\mathcal{F}_K$. Let $\mathcal{G}_K \in A(H)[X]$ be the polynomial

$$\mathcal{G}_K(X) = X^{n_K} + \sum_{i=1}^{n_K} r^G_H(a_{K,i}) \cdot X^{n_K - 1}.$$

Notice that

$$br_K \cdot r^H_K(\mathcal{G}_K(x)) = \mathcal{F}_K\big(br_K \cdot r^H_K(x)\big) = 0. \tag{11.14}$$

Let $\mathcal{G}(X) \in A(H)[X]$ be the polynomial

$$\mathcal{G}(X) = \prod_{K \in \mathcal{P}(A) \cap S(H)} \mathcal{G}_K(X).$$

From relation (11.14) above, it follows that $\mathcal{G}(x) \in \big(\text{Ker } \beta_A\big)(H)$. Since Ker β_A is a nilpotent ideal of A (by to proposition (4.2.1)), it follows that there is an n such that

$$\mathcal{G}^n(x) = \big(\mathcal{G}(x)\big)^n = 0.$$

The relation $\mathcal{G}^n(x) = 0$ expresses the fact that x is integral over $A(G)$ via the map r^G_H.

(3) $\Rightarrow$ (1). Let $H \in S(G)$. Since $A(K)$ is integral over $A(G)$ for all $K \in \mathcal{P}(A)$, it follows that

$$\Big(\prod_{K \in \mathcal{P}(A) \cap S(H)} \overline{A(K)}\Big)^{H} = (T(A))(H)$$

is also integral over $A(G)$. In particular, $(T(A))(H)$ is integral over $A(H)$ as well. $\triangle$

We are now ready to investigate the containments of the second kind in Spec A. We start with the simplest case, namely the containments $P \subseteq \psi(q_G)$, where $q_G \in$ Spec $A(G)$ (see chapter 9 for the definition of the operator ψ_{-}).

(11.10) PROPOSITION.

Let $q_G \in$ Spec $A(G)$, and assume that $P \in$ Spec (A) such that $P(G) \subseteq q_G$. There exists a one-to-one correspondence between the prime ideals of A in $\psi(q_G)$ containing P and the prime ideals of the localized functor $(A/P)_{q_G}$. This correspondence preserves cocharacteristic subgroups and containments of the first kind, second kind, or purely of the second kind.

PROOF. Let $P(G) = p_G \subseteq q_G$. There exists a one-to-one correspondence between the prime ideals of A containing P and the prime ideals of A/P. Under this correspondence, the prime ideals of A containd in $\psi(q_G)$ corespond to prime ideals of A/P contained in $\psi(q_G)/P = \psi(q_G/p_G)$, where $\psi(q_G/p_G)$ is the largest prime ideal of A/P whose value at G is q_G/p_G. By corollary (10.10), there exists a one-to-one correspondence between the prime ideals of A/P contained in $\psi(q_G/p_G)$ and the prime ideals of $(A/P)_{q_G/p_G}$. It is clear that this last Green functor can be canonically identified with $(A/P)_{q_G}$, because $A(G) - q_G$ is disjoint from p_G. It is also clear that both of these correspondences preserve cochararacteristic subgroups and containments. $\triangle$

(11.11) PROPOSITION.

If A is an integral Green functor, then the containment $P \subset \psi(q_G)$ is purely of the second kind if and only if $P \in \phi^{-1}(q_G)$. In particular, if $P \subset Q$ is a containment of the first kind, then $P(G) \subset Q(G)$.

PROOF. Assume that $P \subset \psi(q_G)$ is purely of the second kind, and suppose that $P = A_{(K,\, \mathrm{eq}(\overline{p_K}))}$, for some $\overline{p_K} \in$ Spec $(\overline{A(K)}$. Let $P(G) = p_G \subseteq q_G$. Notice that if $g \in W_G K$, then the inverse image in $A(G)$ of any of the prime ideals ${}^{g}\overline{p_K}$ via the Brauer morphism $br_K \cdot r^G_K$ is exactly p_G. Now if $p_G \subset q_G$, it follows, from the integrality of A and theorem (A2.7), that there exists a prime ideal $\overline{p'_K} \in$ Spec $\overline{A(K)}$ such that $\overline{p_K} \subset \overline{p'_K}$ and the inverse image of $\overline{p'_K}$ in $A(G)$ is exactly q_G. In this case, the ideal $P' = A_{\left(K,\ \mathrm{eq}(\overline{p'_K})\right)}$ is an intermediary prime ideal between P and $\psi(q_G)$ such that $P \subset P'$ is of the first kind. This contradicts the fact that the containment $P \subset \psi(q_G)$ is purely of the second kind.

Conversely, assume that $P \in \phi^{-1}(q_G)$, and assume that P is K-cocharacteristic. If $P = A_{(K,\ \mathrm{eq}(\overline{p_K}))}$, then the inverse image of $\overline{p_K}$ at G is exactly q_G. From theorem $(A2.6)(2)$, it follows that, if $\overline{p'_K}$ strictly contains $\overline{p_K}$, then $P' = A_{\left(K,\ \mathrm{eq}(\overline{p'_K})\right)}$ has the property that $q_G \subset P'(G)$. In particular, such a P' cannot be contained in $\psi(q_G)$. This shows that either $P = \psi(q_G)$ or the containment $P \subset \psi(q_G)$ is purely of the second kind.

Finally assume that the containment $P \subset Q$ is of the first kind. If $p_G = P(G) = Q(G)$, then $P \subset Q \subsetneq \psi(p_G)$. In particular, the containment $P \subset \psi(p_G)$ is not purely of the second kind. However, since $P \in \phi^{-1}(p_G)$, it follows, from the previous argument, that the containment $P \subset \psi(p_G)$ should be purely of the second kind. This contradiction shows that $P(G) \subset Q(G)$. $\triangle$

(11.12) COROLLARY.

Let P be a prime ideal of A and let $P(G) = p_G$. If $P \neq \psi_{p_G}$, then the containment $P \subset \psi_{p_G}$ is of the second kind.

PROOF. Assume that this is not the case and let H be the common cocharacteristic subgroup of both P and ψ_{p_G}. Since $P \neq \psi_{p_G}$ it follows that $0 \neq \psi_0$ in $\left(A/P\right)_{p_G}$. Replace A with the H-characteristic domain $F(A/P) = \left(A/P\right)_{p_G}$. Notice that $A(G)$ is a field. According to proposition (11.10), all the prime ideals of A are H-cocharacteristic. In particular, A is G/H-projective. Hence, A is H-determined, therefore integral. Since $\psi_0 \neq 0$, by (11.11), it follows that $\psi_0(G) \neq 0$. This contradiction shows that the containment $P \subset \psi_{p_G}$ is of the second kind. $\triangle$

(11.13) THEOREM.

(1) Assume that $P \subset Q$ is a containment of the second kind. Then there exists a chain $P \subsetneq P_1 \subset Q$ such that $P \subsetneq P_1$ is a containment of the first kind and $P_1 \subset Q$ is purely of the second kind.

(2) Let $P \subset Q$ be a containment of the second kind. If A is integral, then the above containment is purely of the second kind if and only if $P(G) = Q(G)$.

PROOF. *(1)* Assume that P is K-cocharacteristic. One can easily check that the set of all prime ideals which are K-cocharacteristic and are contained in Q is inductively ordered. It follows, by Zorn lemma, that this set has a maximal element P_1. It is clear that the containment $P_1 \subset Q$ is purely of the second kind.

(2) Assume that $P \subset Q$ is purely of the second kind. Let H be the cocharacteristic subgroup of Q and choose a prime ideal Q' of $A\downarrow_H^G$ such that Q sits over Q'. Since $A\downarrow_H^G /Q'$ is H-bounded, it follows easily that $Q'(L) = A(L)$ for $L < H$. In particular, $Q' = \psi_{Q'(H)} \in \text{Spec }(A\downarrow_H^G)$. From theorem (11.7), it follows that one can choose a prime ideal P' of $A\downarrow_H^G$ such that $P' \subset Q'$ and P sits over P'. Since $P' \subset Q'$ is of the second kind, from proposition (11.11) it follows that $P'(H) = Q'(H)$. Now $P(G) = (r_H^G)^{-1}(P'(H)) = (r_H^G)^{-1}(Q'(H)) = Q(G)$.

Conversely, assume that $P(G) = Q(G)$. Again let Q' be a prime ideal of $A\downarrow_H^G$, such that Q sits over Q', and H is the cocharacteristic subgroup of Q. If $Q = A_{(H,\ \mathrm{eq}(\overline{q(H)}))}$ for some $\overline{q_H} \in \text{Spec }\overline{A(H)}$, it follows, by proposition (11.6), that we may assume $Q' = (A\downarrow_H^G)_{(H,\ \overline{q_H})}$. Choose a prime ideal P' of $A\downarrow_H^G$ such that P sits over P' and $P' \subset Q'$. Notice that $P'(H) \subseteq Q'(H)$. Notice also that

$$P(G) = (r_H^G)^{-1}(P'(H)) = (r_H^G)^{-1}(Q'(H)) = Q(G).$$

Since A is integral, it follows, from theorem (A2.6)*(2)*, that $P'(H) = Q'(H)$. From proposition (11.11), it follows that $P' \subset Q'$ is purely of the second kind. Since P' was an arbitrary prime ideal of $A\downarrow_H^G$ such that P sits over P' and $P' \subset Q'$, we apply theorem (11.7) *(2)* to conclude that $P \subset Q$ is purely of the second kind. $\triangle$

Proposition (11.11) and theorem (11.13) have the following immediate corollary.

(11.14) COROLLARY.

Let A be an integral commutative Green functor.

(1) If $p_G \in \text{Spec } A(G)$, then

$$\dim(\phi^{-1}(p_G)) = \dim_2(\phi^{-1}(p_G)). \tag{11.15}$$

(2) If $P \subset Q$ is a containment of the second kind, then there exists $P_1 \in \text{Spec } A$ such that $P \subseteq P_1 \subset Q$, the containment $P \subseteq P_1$ is of the first kind, and $P_1(G) = Q(G)$.

The corollary (11.14) is false if A is not integral as is shown by the following two examples.

(11.15) EXAMPLE. Let p be a prime number, and let $G = \mathbf{Z}_p$. Let $A(G) = \mathbf{Z}_p$ and $A(1) = \mathbf{Z}_p[X]$. Let $r_1^G : \mathbf{Z}_p \longrightarrow \mathbf{Z}_p[X]$ be the canonical inclusion, and $t_1^G = 0$. Then A is a Green functor. It is clear that all the prime ideals of A are contained in $\phi^{-1}(0)$. If we let $p_1 \subset q_1$ be two prime ideals of $\mathbf{Z}_p[X]$ (for example $p_1 = 0$ and $q_1 = X \cdot \mathbf{Z}_p[X]$), then $P = A_{(1,\, p_1)} \subset A_{(1,\, q_1)} = Q$ is a containment of the first kind in $\phi^{-1}(0)$.

(11.16) EXAMPLE. Again let $G = \mathbf{Z}_p$. Let $A(G) = \mathbf{Z}_p[X]$, and let $A(1)$ be the fraction field of $A(G)$. Let r_1^G be the canonical incusion of $A(G)$ into its fraction field and let t_1^G be zero. Let $q_G = X \cdot \mathbf{Z}_p[X] \in \operatorname{Spec} A(G)$, and let $p_1 = 0 \in \operatorname{Spec} A(1)$. Then $P = 0 = A_{(1,\, p_1)} \subset A_{(G,\, q_G)} = \psi(q_G)$, and the containment $P \subset \psi(q_G)$ is obviously purely of the second kind because $A(1)$ is a field. However, $P(G) = 0 \neq q_G$.

We now come to the main result of this chapter.

(11.17) THEOREM.

Let P and Q be two prime ideals of a Green functor A, and suppose that $P \subset Q$. Assume that the characteristic subgroups of A/P and A/Q are K and H, respectively. Assume that $[K] \neq [H]$. Let q be the characteristic of $A(G)/Q(G)$ and let $n = \log_q|H/K|$ (see corolarry (9.1.7)). Then there exists a chain of prime ideals

$$P = P_0 \subset P_1 \subset \ldots \subset P_n = Q.$$

Moreover, the cocharacteristic subgroups L_i of P_i can be chosen such that $L_i \lhd L_{i+1}$ and $L_{i+1}/L_i = \mathbf{Z}_q$ for $0 \leq i \leq n-1$.

PROOF. We use theorem (11.13) *(1)* to replace P by an K-cocharacteristic prime ideal $\widehat{P}$ such that $P \subseteq \widehat{P} \subset Q$ and the containment $\widehat{P} \subseteq Q$ is purely of the second kind.

Using the Going-Down Restriction Theorem (11.7) we may assume that $H = G$. In this case $Q = \psi_{Q(G)}$. Then $(A/P)_{Q(G)}$ is an K-characteristic domain whose value at K is a W_GK-field. It is enough to show that the Green functor $(A/P)_{Q(G)}$ has a primordial subgroup L such that $K \lhd L$ and $L/K = \mathbf{Z}_q$. Indeed, if L is such a primordial subgroup, then the Green functor $(A/P)_{Q(G)}$ has L-cocharacteristic prime ideals. These prime ideals correspond, by proposition (11.10), to intermediary prime ideals P_1 between P and Q which are L-cocharacteristic. If $L = G$, we are done. Otherwise we may replace P and K with P_1 and L respectively, and continue the argument.

We show that $(A/P)_{Q(G)}$ has a primordial subgroup with the asserted property. Replace A with $(A/P)_{Q(G)}$. Then A is an K-characteristic domain whose value at K is a W_GK-field. Notice also that $G \in \mathcal{P}(A)$. We analyze the map

$$j_{G/K} : A \longrightarrow J_{G/K}(A(K))$$

Let $m_K \in \text{Max } A(K)$. Then $0 = \text{eq}(m_K)$. It folows, from proposition (9.1.10) *(2)*, that

$$J_{G/K}(A(K)) = \left(\text{Inf}_{W_G(K,m_K)}^{N_G(K,m_K)} FP_{A(K)/m_K}\right)\Big\uparrow_{N_G(K,m_K)}^{G}.$$

Since $j_{G/K}(K)$ is onto, it follows easily that $J_{G/K}(A(K))$ is not G/K-projective (otherwise $J_{G/K}(A(K))$ is simple, by proposition (9.1.12), and $j_{G/K}$ is an isomorphism, by corollary (4.1.5); hence A is simple, contradicting the fact that $G \in \mathcal{P}(A)$). Since $J_{G/K}(A(K))$ is not G/K-projective, it follows, by proposition (9.1.12), that the $W_G(K, m_K)$-algebra $A(K)/m_K$ is not projective. Hence q divides the order of $K(K, m_K)$. Let $L \in S(G)$ such that $K \triangleleft L$ and L/K is a subgroup of $K(K, m_K)$ of order q. Then, by proposition (9.1.12), $L \in \mathcal{P}(A)$. $\triangle$

The following corollary is an immediate consequence of proposition (11.10) and theorems (10.18) and (10.20).

(11.18). COROLLARY.

(1) Assume that p_G is a prime ideal in $A(G)$ and let p be the characteristic of $A(G)/p_G$. Let P be a prime ideal in $\phi^{-1}(p_G)$ which is H-characteristic. If $P = A_{(H,\ \text{eq}(\overline{p_H}))}$, then $P = \psi(p_G)$ if and only if p does not divide the order of $K(H, \overline{p_H})$.

The following corollaries are consequences of theorem (11.17).

(11.19) COROLLARY.

Assume that $P \subset Q$ is a containment of the second kind. Assume that P is K-cocharacteristic, Q is H-cocharacteristic, and $K < H$. Let q be the characteristic of $A(G)/Q(G)$. Let $n = \log_q(|H/K|)$. Then there exists a chain of prime ideals

$$P = P_0 \subseteq P_0' \subset P_1 \subseteq P_1' \subset P_2' \subset \ldots \subset P_n \subseteq P_n' = Q$$

such that each containment $P_i \subseteq P_i'$ is of the first kind, and each containment $P_i' \subset P_{i+1}$ is purely of the second kind. Moreover, we may assume that P_i are L_i-characteristic, for some $L_i \in \mathcal{P}(A)$, such that $L_i \triangleleft L_{i+1}$, and $L_{i+1}/L_i \cong \mathbf{Z}_q$.

For $p_G \in \text{Spec } A(G)$, let H be the characteristic subgroup of $\psi(p_G)$, and let p be the characteristic of $A(G)/p_G$. For each $K \in \mathcal{P}(A)$ such that $[H_p] \leq [K]$, let

$$\phi_K^{-1}(p_G) = \{P \in \text{Spec } A \mid P \in \phi^{-1}(p_G) \text{ and } P \text{ is } K - \text{cocharacteristic}\}.$$

With these notations, we have:

(11.20) COROLLARY.

Assume that $p_G \in \operatorname{Spec} A(G)$ *and let* H *be the cocharacteristic subgroup of* ψ_{p_G}. *Then*

$$\dim_2(\phi^{-1}(p_G)) = \max\left(\log_p|H/K| \mid K \in S(G), \text{ such that } \phi_K^{-1}(p_G) \neq \emptyset\right) \leq \log_p|H/H_p|. \tag{11.16}$$

We can now prove the following improvement of corollary (11.14).

(11.21) PROPOSITION.

Let A *be an integral commutative Green functor.*

(1) Let $p_G \in \operatorname{Spec} A(G))$. *Then*

$$\dim_2(\phi^{-1}(p_G)) = \operatorname{ht}_2(\psi(p_G)).$$

(2) Let $P \subset Q$ *be a containment of the second kind. Let* $P(G) = p_G$ *and* $Q(G) = q_G$. *Also let* $S = \{Q' \in \operatorname{Spec} A \mid P \subseteq Q' \subseteq \psi(q_G)\}$. *Then*

$$\dim_2(S) \leq \operatorname{ht}_2(\psi(q_G)) = \dim_2(\phi^{-1}(q_G)). \tag{11.17}$$

In particular,

$$\dim_2(\phi^{-1}(p_G)) \leq \dim_2(\phi^{-1}(q_G)), \tag{11.18}$$

and

$$\max\left(\dim_2(\phi^{-1}(p_G)) \mid p_G \in \operatorname{Spec} A\right) = \dim_2(\phi^{-1}(m_G)) \quad \text{for some } m_G \in \operatorname{Max} A(G). \tag{11.19}$$

(3)

$$\dim(A(G)) \leq \dim(A) \leq \dim(A(G)) + \max\left(\dim_2(\phi^{-1}(m_G)) \mid m_G \in \operatorname{Max} A(G)\right) + 1. \tag{11.20}$$

PROOF. *(1)* According to corollary (9.1.7), we may assume that the integral characteristic of the domain $A(G)/p_G$ is $p > 0$. It is clear that, if $P \in \phi^{-1}(p_G)$, then $P \subseteq \psi(p_G))$. This shows that

$$\dim_2(\phi^{-1}(p_G)) \leq \operatorname{ht}_2(\psi(p_G)).$$

Now let H be the cocharacteristic subgroup of $\psi(p_G)$. Assume that $P \subset \psi(p_G)$ is a containment of the second kind such that P is K cocharacteristic, and K is a minimal primordial subgroup of G for which such a P exists. We apply corollary (11.14) to construct P_1 with the same cocharacteristic subgroup as P such that $P_1 \subset Q$ and $P_1(G) = p_G$. From corollary (11.20), it follows that $\dim_2(\phi^{-1}(p_G)) \geq \log_p(|H/K|)$ where p is the characteristic of the ring $A(G)/p_G$. It is clear from this argument that the reverse inequality follows.

(2) Formula (11.17) is obvious. For (11.18), let $P \in \phi^{-1}(p_G)$, and let $\mathcal{S}_P = \{Q' \in \mathrm{Spec}(A) \mid P \subseteq Q' \subseteq \psi(q_G)\}$. According to (11.17),

$$\dim_2(\mathcal{S}_P) \leq \dim_2(\phi^{-1}(q_G)).$$

Now (11.18) follows from the fact that

$$\dim_2(\phi^{-1}(p_G)) = \max_{P \in \phi^{-1}(p_G)}(\dim_2(\mathcal{S}_P)).$$

Finally, for (11.19), let p_G be a prime ideal of $A(G)$, and let m_G be a maximal ideal of $A(G)$ such that $p_G \subseteq m_G$. Then $P \subseteq \psi(m_G)$ for all $P \in \phi^{-1}(p_G)$. In particular,

$$\dim_2(\phi^{-1}(p_G)) = \mathrm{ht}_2(\psi(p_G)) \leq \mathrm{ht}_2(\psi(m_G)) = \dim_2(\phi^{-1}(m_G)).$$

(3) Let

$$p_{1,G} \subset p_{2,G} \subset \ldots \subset p_{r,G}$$

be a chain of prime ideals of $A(G)$. From the chain

$$\psi(p_{1,G}) \subset \psi(p_{2,G}) \subset \ldots \subset \psi(p_{r,G})$$

it follows that $\dim(A) \geq \dim(A(G))$. Let now

$$P_1 \subset P_2 \subset \ldots \subset P_r$$

be a chain of prime ideals of A. We may of course assume that $P_r = \psi(m_G)$ is a maximal ideal of A. We also assume that at least one of the containments $P_i(G) \subseteq P_{i+1}(G)$ is, in fact, an equality. Hence, suppose that there are exactly $k \leq r$ indices i for which $P_i(G) = P_{i+1}(G)$. Let us label them $i_1 < i_2 < \ldots < i_k$. From theorem (11.13) *(2)*, it follows that the containment $P_{i_j} \subset P_{i_j+1}$ is of the second kind. Hence, the containment $P_{i_j} \subset P_{i_{j+1}}$ is of the second kind as well, for all $1 \leq j \leq k-1$. In particular,

$$P_{i_1} \subset P_{i_2} \subset \ldots \subset P_{i_k}$$

is a chain of the second kind of prime ideals contained in $P_r = \psi(m_G)$. Therefore, $k - 1 \leq \mathrm{ht}_2(\psi(m_G)) = \dim_2(\phi^{-1}(m_G))$. Now since all the $r-k$ ideals $P_i(G)$, for $i \notin \{i_1, i_2, \ldots, i_k\}$, are distinct, and form a chain in Spec $A(G)$, it follows that $r-k-1 \leq \dim(A(G))$. From these inequalities, we obtain that

$$r-1=(r-k-1)+(k-1)+1 \leq \dim(A(G))+\dim_2(\phi^{-1}(m_G))+1.$$

The inequality (11.20) follows immediately from the above inequality. $\triangle$

(11.22) PROPOSITION.

Let A be an integral Green functor and let $p_G \in$ Spec $A(G)$. Suppose that P is a prime ideal of A, $P \in \phi^{-1}(p_G)$. Then the set

$$\mathcal{S}=\{Q \in \operatorname{Spec}(A) \mid P \subseteq Q \subseteq \psi_{p_G}\}$$

is finite.

PROOF. Let H be the cocharacteristic subgroup of P and let p be the integral characteristic of $A(G)/p_G$. We assume $p>0$ (otherwise $\mathcal{S}=\{\psi_{p_G}\}$). Let $Q \in \mathcal{S}$. Suppose that $P \neq Q$. Since $Q(G)=P(G)$, it follows, by proposition (11.11), that the containment $P \subset Q$ is of the second kind. By theorem (11.17), it follows that there exists $P_1 \in$ Spec (A) such that $P \subset P_1 \subseteq Q$ and P_1 is L-cocharacteristic, for some L such that $H \triangleleft L$ and $L/H=\mathbf{Z}_p$. It is clear that $P_1 \in \mathcal{S}$. We show that there exists only finitely many L-cocharacteristic prime ideals in $\mathcal{S}$. Let $\widetilde{P}$ be such a prime ideal, and let $\widetilde{P}'$ be a prime ideal of $A\downarrow_L^G$ such that $\widetilde{P}$ sits over $\widetilde{P}'$. Let $\{P_i'\}_{i=1}^m$ be all the prime ideals of $A\downarrow_L^G$ such that P sits over P_i', for $i=1,\ 2,\ \ldots\ m$. It follows, by theorem (11.7), that $\widetilde{P}$ contains one of the prime ideals P_i'. Moreover, since A is integral, it follows that $P_i'(H)=\widetilde{P}'(H)$. Since $\widetilde{P}=(\downarrow_L^H)^{-1}(\widetilde{P}')$, it follows that

$$\{\widetilde{P} \in \mathcal{S} \mid \widetilde{P} \text{ is } L-\text{cocharacteristic}\}$$

has at most m elements. It follows, by induction on the size of L, that $\mathcal{S}$ has finitely many elements. $\triangle$

(11.23) COROLLARY.

Assume that A is a commutative Green functor such that $\mathcal{P}(A)$ has the following property:

-whenever $K \subset H$ are two primordial subgroups of A, then either K is not normal in H, or K/H is not a cyclic group of order p, for some prime number p which is not invertible in $A(G)$.

Then A is totally decomposable.

PROOF. We check that A satisfies the condition *(1)* from proposition (7.4.8). Let $K \in \mathcal{P}(A)$. Denote by $A_K = J_{G/K}(\overline{A(K)})$. Notice that A_K is an integral Green functor which is K characteristic. We show that A_K is G/K-projective. Assume that this is not the case. Let H be a maximal subgroup in $\mathcal{P}(A_K)$. Since $[K] < [H]$, we may assume that $K < H$. Let $Q = A_{(H,\ \mathrm{eq}(\overline{q_H}))}$ be an H-cocharacteristic prime ideal of A_K. Let $q_G = Q(G)$. Since the extension

$$A_K(G) \xrightarrow{r_H^G} A_K(H)$$

is integral, and the map r_H^G is injective, it follows that there exists a prime ideal p_K of $A_K(K)$ such that $(r_H^G)^{-1}(p_K) = q_G$. But this shows that both prime ideals Q and $P = A_{(K,\ \mathrm{eq}(\overline{p_K}))}$ are in $\phi^{-1}(q_G)$. From the maximality of H, it follows that the cocharacteristic subgroup of the ideal of $\psi(q_G)$ is H. Since $P \subset \psi(q_G)$, it follows, by theorem (11.17), that there exists a chain of subgroups

$$K = L_0 \triangleleft L_1 \triangleleft \dots . \triangleleft L_n = H$$

such that $L_i \in \mathcal{P}(A_K) \subseteq \mathcal{P}(A)$, and $L_{i+1}/L_i = \mathbf{Z}_q$, where q is the characteristic of $A(G)/q_G$. But this contradicts the assumption on $\mathcal{P}(A)$. In conclusion, the only primordial subgroups of A_K are conjugate to K, hence $A_K = J_{G/K}(\overline{A(K)})$ is G/K-projective. $\triangle$

We conclude this chapter with an example.

(11.24) EXAMPLE. Let B be the Burnside ring functor for G (over R). Since W_GH acts trivially on $\overline{B(H)}$ it follows that the prime ideals of B are of the form $B_{(H,\ P)}$, where P is a prime ideal of R. Moreover $B_{(H,\ P)}$ is maximal if and only if P is maximal and the characteristic of the residual field R/P does not divide the order of $|W_GH|$. Let p and q be the integral characteristics of R/P and R/Q respectively. It can be shown that $B_{(K,\ P)} \subset B_{(H,\ Q)}$ if and only if $[K] \le [H] \le [H^q]$ and $P \subseteq Q$. It follows that, if

$$k = \max \{\log_p(|H^p/H|) \mid H \in S(G) \text{ and } p \text{ prime, not invertible in } R\},$$

then the Krull dimension of B is $\dim(B) = \dim(R) + k$.

12. Characteristic Subgroups and Essential Submodules.

Throughout this chapter, A is a Green functor (not necessarily commutative) and M is a left-A-module. In this chapter, we introduce the notion of a characteristic subgroup of M and we prove some of its properties.

(12.1) DEFINITION. A subgroup $H \in S(G)$ is called *characteristic* for M if M has a submodule N (not necessarily proper) which is H-characteristic. The set of characteristic subgroups of M is denoted with $\mathrm{Char}(M)$. Notice that $\mathrm{Char}(M)$ is closed under conjugation, and, if $N \subseteq M$, then $\mathrm{Char}(N) \subseteq \mathrm{Char}(M)$.

(12.2) EXAMPLE. Let M be an H-characteristic Mackey functor and let $0 \neq N \subseteq M$. From proposition (6.1.3) *(1)*, it follows that N is H-characteristic as well. In particular, $\mathrm{Char}(M) = [H]$.

Notice that $\mathrm{Char}(M) \subseteq \mathcal{P}(A)$. However, it is not true in general that $\mathrm{Char}(M) \subseteq \mathcal{P}(M)$.

(12.3) EXAMPLE. Let $G = \mathbf{Z}_2$, and let $M(H) = \mathbf{Z}_2$ for $H \in S(G)$. Suppose that r_1^G is the zero map and t_1^G is the identity of $\mathbf{Z}_2$. One can check easily that M is a Mackey functor, and that $\mathcal{P}(M) = \{1\}$, but $\mathrm{Char}(M) = \{G\}$.

In this chapter we show that if M is non-zero then $\mathrm{Char}(M)$ is non-empty. In particular M has a submodule N (not necessarily proper) which is characteristic. We also show that the internal sum of all the characteristic submodules of M is essential in M. If $\mathrm{Char}(M) = [H]$ we refer to M as *H-cospecial.* If N is a submodule of M such that M/N is H-cospecial we refer to N as H-special in M. We show that if A and M satisfy certain noetherianity conditions then every submodule of M can be written as a finite intersection of submodules which are special in M. We also prove an induction theorem for the H-cospecial modules. The notions and results from this chapter are fundamental for our approach to tertiary submodules of M which is done in chapter 13.

Let $H \in S(G)$. Let:

$$X_G(H) = \coprod_{K<H} G/K,$$

$$\Omega_H(M) = \mathrm{Ker}\left(\theta_M^{X_G(H)} : M \longrightarrow M_{X_G(H)}\right).$$

Also let

$$\mathrm{Supp}_\Omega(M) = \{H \in S(G) \mid \Omega_H(M) \neq 0\}.$$

Finally, let

$$\mathrm{C}_H(M) = \{N \leq M \mid N \text{ is } H-\text{characteristic}\},$$

and

$$\mathcal{C}(M) = \bigcup_{H \in \mathrm{Char}(M)} \mathcal{C}_H(M) = \{N \leq M \mid N \text{ is characteristic}\}.$$

For the above notation, we adopt the convention that $X_G(1) = \emptyset$ and $M_\emptyset = 0$; hence $\Omega_1(M) = M$. In particular, if M is non-zero, then $1 \in \mathrm{Supp}_\Omega(M)$; hence $\mathrm{Supp}_\Omega(M) \neq \emptyset$.

(12.4) PROPOSITION.

Let M be a non-zero left-A-module.

(1) If N is an H-bounded submodule of M, then $N \leq \Omega_H(M)$. Moreover, if $N \neq 0$, then $H \in \mathrm{Supp}_\Omega(M)$. In particular, $\mathrm{Char}(M) \subseteq \mathrm{Supp}_\Omega(M)$.

(2) $\mathrm{Max}\,(\mathrm{Supp}_\Omega(M)) = \mathrm{Max}\,(\mathrm{Char}(M))$. *Moreover, if $H \in \mathrm{Max}\,(\mathrm{Char}(M))$, then $\Omega_H(M)$ is H-characteristic, and $N \leq \Omega_H(M)$ whenever $N \in \mathcal{C}_H(M)$.*

(3) $\mathrm{Char}(M) \neq \emptyset$, *and* $\mathcal{C}(M) \neq \emptyset$.

PROOF. *(1)* If N is H bounded, then $N(K) = 0$, for all $K < H$. But this is equivalent to $N \leq \Omega_H(M)$. Of course, if $N \neq 0$, then $\Omega_H(M) \neq 0$; hence $H \in \mathrm{Supp}_\Omega(M)$. The containment $\mathrm{Char}(M) \subseteq \mathrm{Supp}_\Omega(M)$ is now obvious.

(2) If $H \in \mathrm{Max}\,(\mathrm{Char}(M))$, then, from *(1)*, it follows that $H \in \mathrm{Supp}_\Omega(M)$. Conversely, let $H \in \mathrm{Max}\,(\mathrm{Supp}_\Omega(M))$. We show that $\Omega_H(M)$ is H-characteristic. Indeed, if not, let N be the kernel of the map

$$\Omega_H(M) \longrightarrow (\Omega_H(M))_{G/H},$$

and let L be a minimal primordial subgroup of N. Then N is non-zero, L-bounded, and $H < L$. In particular, $L \in \mathrm{Supp}_\Omega(M)$. This contradicts the maximality of H in $\mathrm{Supp}_\Omega(M)$.

(3) Follows immediately from *(2)* because the set $\mathrm{Supp}_\Omega(M)$ is non-empty. Hence, it must have maximal elements. $\triangle$

Due to the importance of proposition (12.4) *(3)*, we prefer to restate it as follows.

(12.5) COROLLARY.

If M is a non-zero left-A-module, then M has a submodule which is characteristic.

(12.6) DEFINITION. Let M be a left-A-module. The submodule

$$\operatorname{chsoc}(M) = \sum_{N \in C(M)} N$$

is called *the characteristic socle* of M.

For the connection with the notion of socle from classical algebra, we first give the following definition.

(12.7) DEFINITION. Let M be a left-A-module and N be a non-zero submodule of M. Then N is called *essential* in M if, whenever N' is a non-zero submodule of N, $N \cap N' \neq 0$.

(12.8) PROPOSITION.

Let M be a non-zero left-A-module. Then $\operatorname{chsoc}(M)$ *is essential in M.*

PROOF. According to corollary (12.5), every non-zero submodule of M has a characteristic submodule. This must be non-zero and contained in $\operatorname{chsoc}(M)$. Δ

The notion of characteristic subgroups of a left-A-module M can be seen as the dual notion to primordial subgroups. We have the following result.

(12.9) PROPOSITION.

Let $X = \coprod_{H \in \mathrm{Char}(M)} G/H$. The map

$$M \longrightarrow M_X$$

is injective.

PROOF. Assume that this is not the case, and let N be the kernel of the above map. Also let N' be a characteristic submodule of N. Since $\mathrm{Char}(N') \subseteq \mathrm{Char}(M)$, it follows that $N' \in \mathrm{C}_H(M)$ for some $H \in \mathrm{Char}(M)$. However, from the above map one can immediately see that $N(H) = 0$ whenever $H \in \mathrm{Char}(M)$. In particular, $N'(H) = 0$, contradicting the fact that N' is H-characteristic. Δ

An important special type of left-A-modules M are the ones for which any two characteristic subgroups are conjugate.

(12.10) THEOREM.

The following assertions are equivalent:

(1) $\mathrm{Char}(M) = [H]$ *for some* $H \in \mathcal{P}(A)$.

(2) M *has a characteristic submodule which is essential.*

When these equivalent conditions are satisfied, $\mathrm{chsoc}(M) = \Omega_H(M)$ *and* $\mathrm{chsoc}(M)$ *is* H*-characteristic. Moreover, the map*

$$M \longrightarrow M_{G/H}$$

is injective.

PROOF. (1) $\Rightarrow$ (2). Since $H \in \mathrm{Max}\,(\mathrm{Supp}_\Omega(M))$ it follows, from *(2)* of proposition (12.4), that $\Omega_H(M)$ is H-characteristic. Let N be a non-zero submodule of M, and let N' be a characteristic submodule of N. Then N' must be H-characteristic; hence, from *(1)* of proposition (12.4), $N' \subseteq \Omega_H(M)$. This shows that $\Omega_H(M)$ is essential and characteristic; hence (2).

(2) $\Rightarrow$ (1). Let N be an essential submodule of M which is H-characteristic. Assume that $K \in \mathrm{Char}(M)$, and let $N' \in \mathcal{C}_K(M)$. Then $N \cap N' \neq 0$, and $N \cap N'$ is a submodule of both N and N'. From proposition (6.1.3) *(1)*, it follows that $N \cap N'$ must be both H- and K-characteristic. This shows that $[H] = [K]$.

It is clear that, in this case, $\mathrm{chsoc}(M) = \Omega_H(M)$. Finally, the injectivity of the map $M \longrightarrow M_{G/H}$ follows from proposition (12.9). $\triangle$

(12.11) DEFINITION. A left-A-module with the above property is called *cospecial.* We sometimes refer to these modules as being *H-cospecial.* If M is a left-A-module and N is a proper submodule of M, then N is called *special* in M if M/N is cospecial. When M/N is H-cospecial, we refer to N as being *H-special* in M.

For examples of H-cospecial module which are not H-characteristic, see (12.3) and (13.11).

(12.12) DEFINITION. (1) A left-A-module M is called *coirreducible*, if whenever N and N' are non-zero submodules of M, then $N \cap N' \neq 0$.

(2) Let M be a left-A-module and N be a proper-submodule of M. N is called *irreducible* if M/N is coirreducible.

(12.13) PROPOSITION.

Let M be a left-A-module, and let N be a irreducible submodule of M. Then N is special in M.

PROOF. Choose a submodule N' containing N such that N'/N is a characteristic submodule of M/N. Since M/N is coirreducible, it follows immediately that N'/N is essential in M/N. In particular, M/N satisfies *(2)* of theorem (12.10), hence M/N is cospecial. $\triangle$

We now record some properties of $\mathrm{Char}(M)$.

(12.14) PROPOSITION.

(1) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of left-A-modules. Then

$$\mathrm{Char}(L) \subseteq \mathrm{Char}(M) \subseteq \mathrm{Char}(L) \cup \mathrm{Char}(N).$$

(2) If M_i are left-A-modules then

$$\mathrm{Char}(\bigoplus_{i\in I} M_i) = \bigcup_{i\in I} \mathrm{Char}(M_i).$$

PROOF. *(1)* The containment $\mathrm{Char}(L) \subseteq \mathrm{Char}(M)$ is obvious. Let $H \in \mathrm{Char}(M)$, and let $N' \in \mathcal{C}_H(M)$. If $N' \cap L \neq 0$, then $N' \cap L$ is an H-characteristic submodule of L; hence $H \in \mathrm{Char}(L)$. On the other hand, if $N' \cap L = 0$, then N' is isomorphic to a submodule of N; hence $H \in \mathrm{Char}(N)$.

(2) From *(1)*, it follows that

$$\bigcup_{i\in I} \mathrm{Char}(M_i) \subseteq \mathrm{Char}(\bigoplus_{i\in I} M_i).$$

Conversely, choose $H \in \mathrm{Char}(\bigoplus_{i\in I} M_i)$, and let M be an H-characteristic submodule of $\bigoplus_{i\in I} M_i$. Since the non-zero submodules of M are still H-characteristic, we may assume that M is generated by a non-zero element of $M(H)$; i.e. $M = A\langle m_H\rangle$, for some $m_H \neq 0$ in $\bigoplus_{I\in I} M_i(H)$. This element m_H can be written as

$$m_H = \sum_{i\in I} m_{i,H}, \qquad \text{for some } m_{i,H} \in M_i(H)$$

such that all $m_{i,H}$ are zero except for a finite set of indices J. Since $M = A\langle m_H\rangle \subseteq \sum_{i\in J} M_i$, we may assume that the set I of indices is finite. Now the reverse containment follows inductively from *(1)*. $\triangle$

(12.15) COROLLARY.

Assume that $H \in S(G)$, and that $(N_i)_{i\in I}$ is a set of H-special submodules of M. Then $\bigcap_{i\in I} N_i$ is also H-special.

PROOF. Let $N = \bigcap_{i\in I} N_i$. We apply proposition (12.14) to the injective map

$$\frac{M}{N} = \frac{M}{\bigcap_{i\in I} N_i} \longrightarrow \bigoplus_{i\in I} \frac{M}{N_i}$$

to conclude that

$$\mathrm{Char}(M/N) \subseteq \mathrm{Char}\Big(\bigoplus_{i\in I}(M/N_i)\Big) = \bigcup_{i\in I} \mathrm{Char}(M/N_i) = [H].$$

Now the assertion follows from the fact that $\mathrm{Char}(M/N)$ is non-empty and closed under conjugation. Δ

(12.16) DEFINITION. Let M be a left-A-module. A *special decomposition* of a submodule L of M is obtained by writing L as a finite intersection $L = M_1 \cap \ldots \cap M_n$ of special submodules M_i of M, so that

(i) the decomposition is irredundant,

(ii) $\mathrm{char}(M/M_i) \neq \mathrm{char} M/M_j)$ for $i \neq j$.

(12.17) THEOREM (Special Decomposition Theorem).

Every submodule L of a left-noetherian A-module M has a special decomposition. If $L = M_1 \cap \ldots M_m = N_1 \cap \ldots N_n$ are two special decompositions of L in M, then $m = n$ and

$$\{\mathrm{char}(M/M_i)\} = \{\mathrm{char}(M/N_j)\} = \mathrm{Char}(M/L).$$

PROOF. See the proof of the tertiary decomposition theorem for finitely generated modules over left-noetherian rings from [St] p. 160-162. Δ

(12.18) COROLLARY.

Let R be a commutative noetherian ring and let M be a finitely generated Mackey functor for G (over R). If $L \subset M$, then L has a special decomposition, and any two special decompositions of L satisfy the property from theorem (12.17).

PROOF. Since R is noetherian, it follows that the Burnside ring Green functor over R is totally noetherian. Since M is finitely generated, we conclude that M is noetherian (in fact, totally noetherian). Now the assertion follows from theorem (12.17). Δ

We conclude this chapter with an induction theorem. Notice that, if A is a Green functor, then $\mathrm{Char}(A) \subseteq \mathcal{P}(A)$. The above containment is not an equality in general as is shown by the following example.

(12.19) EXAMPLE. Let p be a prime number, and let G be a p-group. Assume that S is any $\mathbf{Z}_p$ algebra on which G acts trivially. Then it is clear that the Green functor $A = FP_S$ is 1 characteristic; hence $\mathrm{Char}(A) = \{1\}$. However, $\mathcal{P}(A) = S(G)$.

If M is a left-A-module, then it is not true in general that $\mathrm{Char}(M) \subseteq \mathrm{Char}(A)$. Indeed, in the previous example, we have seen that $\mathrm{Char}(A)$ may be smaller then $\mathcal{P}(A)$. For such a Green functor, choose $H \in \mathcal{P}(A) - \mathrm{Char}(A)$. Then H is characteristic for the left-A-module $J_{G/H}(\overline{A(H)})$, but it is not characteristic for A.

We notice that, although $\mathrm{Char}(A)$ is smaller then $\mathcal{P}(A)$, we can bound $\mathcal{P}(A)$ in terms of $\mathrm{Char}(A)$.

(12.20) THEOREM.

Let Π *be the set of prime numbers* $p \mid |G|$ *such that* $p \cdot 1_{A(G)}$ *is not a unit in* $A(G)$. *Then*

$$\mathcal{P}(A) \subseteq \mathcal{D}(A) \subseteq \mathcal{H}_{\Pi}^{G}\mathrm{Char}(A).$$

PROOF. Immediate consequence of propositions (12.9), (4.1.7) and the Dress Induction Theorem (2.9). $\triangle$

Let M be an H-cospecial left-A-module. In order to obtain a good induction theorem for M, in light of theorem (2.5), it is enough to derive a good induction theorem for the Green functor $A/\mathrm{Ann}_A(M)$. Here is our result in this direction.

(12.21) THEOREM (Induction Theorem for Cospecial Modules).

Assume that $\overline{A(K)}$ *is left-noetherian for all* $K \in \mathcal{P}(A)$, *and that* M *is* H*-cospecial. Then*

(1) If $(\mathrm{chsoc}(M))(H)$ *has no torsion elements, then* M *is* H*-characteristic. In particular,* M *satisfies induction theorem* (6.2.5).

(2) Assume that $(\mathrm{chsoc}(M))(H)$ *is* p *torsion, where* p *is a prime number. If* M *is left-noetherian then* M *satisfies induction theorem* (6.2.4).

PROOF. The proof is achieved in two steps. At step I we study the minimal primordial subgroups of M. Let $K \in \mathrm{Min}\,(\mathcal{P}(M))$. We show that there exists a prime ideal P of A which is K-cocharacteristic and a prime ideal Q of A which is H-cocharacteristic such that

$P \subseteq Q$. At step II we use theorems (6.2.4), (6.2.5) and corollary (9.1.7) to prove *(1)* and *(2)*.

STEP I. Notice that the map $M \longrightarrow M_{G/H}$ is injective. Let $K \in \text{Min}\,(\mathcal{P}(M))$. We know that the only subgroups $L \in S(G)$ for which the submodule

$$A\langle M(K)\rangle = \sum_{m \in M(K)} A\langle m\rangle$$

can be non-zero are the subgroups for which $[K] \leq [L]$. Since $A\langle M(K)\rangle \cap \text{chsoc}(M)$ is a non-zero submodule of the H-characteristic submodule chsoc(M), it follows that $\Big(A\langle M(K)\rangle \cap \text{chsoc}(M)\Big)(H) \neq 0$. In particular, $\big(A\langle M(K)\rangle\big)(H) \neq 0$; hence $[K] \leq [H]$. We may assume that $K \leq H$.

Notice that $M(K)$ is a left-$\overline{A(K)}$-module. Choose a tertiary submodule $M_{1,K}$ of $M(K)$ (see [St] p. 160-162), and let $\text{ass}_{\overline{A(K)}}(M_{1,K}) = \overline{p_K}$ for some prime ideal $\overline{p_K}$ of $\overline{A(K)}$. Let

$$M_1 = A\langle \sum_{g \in W_G K} {}^g M_{1,K}\rangle = \sum_{g \in W_G K} \; \sum_{m \in {}^g M_{1,K}} A\langle m\rangle$$

Notice that $M_1(K) = \sum_{g \in W_G K} {}^g M_{1,K}$. Hence

$$\text{Ann}_{\overline{A(K)}}(M_1(K)) = \text{eq}(\overline{p_K}).$$

One can easily see that

$$\text{Ann}_A(M_1) = A_{(K,\ \text{Ann}_{\overline{A(K)}}\ M_1(K))} = A_{(K,\ \text{eq}(\overline{p_K}))} = P$$

is a prime ideal of A. Let $N = M_1 \cap \text{chsoc}(M) \neq 0$. In particular, $N(H) \neq 0$ and N is H-characteristic. Now notice that

$$P = \text{Ann}_A(M_1) \subseteq \text{Ann}_A(N) = A_{(H,\ \text{Ann}_{\overline{A(H)}}\ N(H))}. \tag{12.2}$$

Since $\overline{A(H)}$ is left-noetherian, it follows that $N(H)$ has an associated prime ideal $\overline{q_H}$. Notice that $\overline{q_H} \in \text{Ass}_{\overline{A(H)}}\Big(\big(\text{chsoc}(M)\big)(H)\Big)$. In particular, $\text{Ann}_{\overline{A(H)}}\ N(H) \subseteq \overline{q_H}$. Since $N(H)$ is an $R[W_G H]$-module, it follows that

$$\text{Ann}_{\overline{A(H)}}\ N(H) \subseteq \text{eq}(\overline{q_H}).$$

Since the operator $A_{(H,\ -)}$ preserves containments, it follows that

$$A_{(H,\ \text{Ann}_{\overline{A(H)}}\ N(H))} \subseteq A_{(H,\ \text{eq}(\overline{q_H}))} = Q, \tag{12.3}$$

where Q is an H-cocharacteristic prime ideal of A. From containments (12.2) and (12.3), it follows that $P \subseteq Q$.

STEP II. *(1)* If $(\mathrm{chsoc}(M))(H)$ has no torsion, it follows that $\overline{q_H}$ is a prime ideal of $\overline{A(H)}$ such that the prime ring $\overline{A(H)}/\overline{q_H}$ has characteristic zero. In particular, the prime Green functor A/Q has characteristic zero. Since $P \subseteq Q$, it follows, from corollary (9.1.7), that $K = H$. This shows that $M = \mathrm{chsoc}(M)$. Hence M is H-characteristic and M satisfies induction theorem (6.2.5).

(2) Since $(\mathrm{chsoc}(M))(H)$ has only p torsion, it follows that the prime ring $\overline{A(H)}/\overline{q_H}$ has characteristic p. Since A/Q has characteristic p, and $P \subseteq Q$, it follows, from corollary (9.1.7), that $[H] \leq [K^p]$. Therefore $[H_p] \leq [K]$. In order to prove that M satisfies induction theorem (6.2.4) it is enough to show that $p^s \in \mathrm{Ann}_A(M)$ for some $s \geq 1$. We first prove that for some $n \geq 1$, p^n annihilates $(\mathrm{chsoc}(M))(H)$. We then show that for some $s \geq 1$, p^s annihilates M.

Notice first of all that, since M is left-noetherian, it follows that $\mathrm{chsoc}(M)$ is left noetherian as well. In particular, $(\mathrm{chsoc}(M))(H)$ is an $W_G H$ left-noetherian $\overline{A(H)}$ module. Hence, $(\mathrm{chsoc}(M))(H)$ is finitely generated, thus there exists $n \geq 1$ such that $p^n \in \mathrm{Ann}_{\overline{A(H)}}(\mathrm{chsoc}(M))(H)$. This shows that $p^n \in \mathrm{Ann}_A(\mathrm{chsoc}(M))$.

For $k \geq 1$, let

$$M_k = \{m \in M \mid (p^n)^k \cdot A\langle m\rangle(H) \subset (\mathrm{chsoc}(M))(H)\}.$$

It is easy to check that M_k is a left submodule of M. Indeed, M_k is just the counterimage of the left submodule $\mathrm{chsoc}(M)$ of M via the map

$$M \xrightarrow{(p^n)^k} M,$$

which is obviously functorial. Since M is left noetherian, the chain

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_k \subseteq \ldots$$

must be stationary. Assume that $M_k = M_{k+1}$. We show that $p^{n(k+1)} \cdot M = 0$. Indeed, otherwise $(p^{n(k+1)} \cdot M) \cap \mathrm{chsoc}(M) \neq 0$. This means that

$$(p^{n(k+1)} \cdot M(H)) \cap (\mathrm{chsoc}(M))(H) \neq 0.$$

In particular, there exists $m_H \in M(H)$ such that

$$p^{n(k+1)} \cdot m_H \in (\mathrm{chsoc}(M))(H) \quad \text{and} \quad p^{n(k+1)} \cdot m_H \neq 0.$$

However, it is easy to see that $m_H \in M_{k+1}(H) = M_k(H)$; hence $p^{nk} \cdot m_H \in \big(\mathrm{chsoc}(M)\big)(H)$. Since $p^n \in \mathrm{Ann}_A\big(\mathrm{chsoc}(M)\big)$, it follows that $p^{n(k+1)} \cdot m_H = 0$. This contradiction shows that $p^{n(k+1)} \in \mathrm{Ann}_A(M)$. $\triangle$

13. Tertiary and Primary Submodules.

In this chapter we investigate the tertiary and primary submodules of a left-A-module M. Throughout this chapter, A is a Green functor such that $\overline{A(H)}$ is left-noetherian for all $H \in \mathcal{P}(A)$. We give characterization theorems for cotertiary and coprimary modules. From commutative algebra, we know that the tertiary submodules of a finitely generated module over a commutative noetherian ring are primary. It follows, as a consequence of our characterization theorems, that the analogue result does not hold in the appropiate Mackey functor setting. We prove that the tertiary decomposition theorem holds in the appropriate setting. We also show that the primary decomposition theorem does not hold in general (fact previously noticed by Lewis). We also give induction theorems for cotertiary and coprimary modules.

(13.1) DEFINITION. Let M be a non-zero left-A-module. A two-sided ideal I is *associated* to M if there exists a non-zero submodule N of M such that $\mathrm{Ann}_A(N') = I$ for all non-zero submodules N' of N. It follows, from (12.5), that N can be chosen to be characteristic. Moreover, we assume that N is generated by one element $n_H \in N(H)$, where $H \in \mathrm{Min}\,(\mathcal{P}(N))$. It is clear that such a submodule has the property that $\mathcal{P}(N) = \mathrm{Char}(N) = [H]$. The ideal I must be prime, for if $J \cdot J' \subseteq I$, and $J' \not\subseteq I$, it follows that $J' \not\subseteq \mathrm{Ann}_A(N)$. In particular, $J' \cdot N$ is a non-zero submodule of N. But then $J \subseteq \mathrm{Ann}_A(J' \cdot N) = \mathrm{Ann}_A(N) = I$; hence I is prime. We denote the set of all prime ideals associated to M by $\mathrm{Ass}_A(M)$.

Up to this point, our considerations are applicable to any Green functor A. We now use the assumption that $\overline{A(H)}$ is left-noetherian for all $H \in \mathcal{P}(A)$ to prove the following result.

(13.2) PROPOSITION.

(1) If M is a non-zero module, then $\mathrm{Ass}_A(M) \neq \emptyset$.

(2) If $P \in \mathrm{Ass}_A(M)$ then there exists a characteristic submodule of M such that $P = \mathrm{Ann}_A(N)$. In this case, if P is H-cocharacteristic, then N is H-characteristic.

(3) $\mathrm{Char}(M) = \{H \in S(G) \mid \exists\, P \in \mathrm{Ass}_A(M)$ such that P is H-cocharacteristic$\}$.

PROOF. *(1)* Let N be an H-characteristic submodule of M. Then $N(H)$ is a left-$\overline{A(H)}$-module. Since $\overline{A(H)}$ is left-noetherian, it follows that the set

$$\mathcal{F} = \{\mathrm{Ann}_{\overline{A(H)}}(N'_H) \mid N'_H \text{ non}-\text{zero } W_G H-\text{invariant submodule of } N(H)\}$$

has maximal elements. Let $\overline{P_H}$ be a maximal element of $\mathcal{F}$. It follows, from theorem (A1.22), that $\overline{P_H} = \mathrm{eq}(\overline{p_H})$ for some prime ideal $\overline{p_H}$ of $\overline{A(H)}$. Let $P = A_{(H,\overline{P_H})}$. We show that $P \in \mathrm{Ass}_A(M)$. Let N'_H be a $W_G H$-invariant submodule of $N(H)$ such that $\mathrm{Ann}_{\overline{A(H)}}(N'_H) = \overline{P_H}$. Let

$$N' = A\langle N'_H\rangle = \sum_{n_H \in N'_H} A\langle n_H\rangle.$$

Then $\mathrm{Ann}_A(N') = P$. Moreover, N' is H-characteristic. It follows easily, from the maximality of $\overline{P_H}$ in $\mathcal{F}$, that $\mathrm{Ann}_A(N") = P$ for all non-zero submodules $N"$ of N'. Hence $P \in \mathrm{Ass}_A(M)$.

(2)-(3) Obvious. $\triangle$

We record the following classical properties of Ass:

(13.3) PROPOSITION.

If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *is an exact sequence of left A modules then*

$$\mathrm{Ass}_A(L) \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(L) \cup \mathrm{Ass}_A(N).$$

(2) If M_i *are left-A-modules then* $\mathrm{Ass}_A(\bigoplus_{i\in I} M_i) = \bigcup_{i\in I} \mathrm{Ass}_A(M_i)$.

(3) If N *is an irreducible submodule of* M, *then* $\mathrm{Ass}_A\big(M/N\big)$ *has only one member.*

(4) If P *is a prime ideal of* A, *then* $\mathrm{Ass}_A(A/P) = \{P\}$.

PROOF. See [St] p. 160-162. $\triangle$

(13.4) DEFINITION. A non-zero left A module M is called *cotertiary* if $\mathrm{Ass}_A(M)$ has exactly one member; this element is denoted by $\mathrm{ass}_A(M)$. If $\mathrm{ass}_A(M) = P$, we refer to M as being P-cotertiary. If M is a left A module and N is a submodule of M, then N is *tertiary in M* if M/N is cotertiary. When $\mathrm{ass}_A\big(M/N\big) = P$ we refer to N as P-tertiary in M.

(13.5) EXAMPLE. From proposition (13.3), we conclude that, if N is an irreducible submodule of M, then N is tertiary in M. Moreover, if P is a prime ideal of A, then P is P-tertiary in A.

(13.6) PROPOSITION.

(1) Every intersection of P-tertiary submodules of M is P-tertiary in M.

(2) If M is a left-noetherian A-module then $\mathrm{Ass}_A(M)$ is finite.

PROOF. *(1)* See [St] p. 160-162.

(2) Since $\mathrm{chsoc}(M)$ is essential in M, it follows that $\mathrm{Ass}_A(\mathrm{chsoc}(M)) = \mathrm{Ass}_A(M)$. Now since $\mathrm{chsoc}(M)$ is noetherian, it follows that

$$\mathrm{chsoc}(M) = \sum_{i=1}^{n} N_i$$

for a finite collection $\{N_i\}_{i=1}^{n}$ of characteristic submodules of M. Since

$$\mathrm{Ass}_A(\mathrm{chsoc}(M)) = \mathrm{Ass}_A(\sum_{i=1}^{n} N_i) \subseteq \mathrm{Ass}_A(\bigoplus_{i=1}^{n} N_i) = \bigcup_{i=1}^{n} \mathrm{Ass}_A(N_i),$$

it follows that it is enough to prove *(2)* when M is an H-characteristic left-noetherian A-module. It follows easily, from theorem (A1.29), that $P \in \mathrm{Ass}_A(M)$ if and only if $P = A_{(H,\,\mathrm{eq}(\overline{p_H}))}$ for some $\overline{p_H} \in \mathrm{Ass}_{\overline{A(H)}}(M(H))$. Since $M(H)$ is a left-noetherian $\overline{A(H)}$-module, we conclude that the set $\mathrm{Ass}_{\overline{A(H)}}(M(H))$ is finite. Hence $\mathrm{Ass}_A(M)$ is finite as well. $\triangle$

(13.7) DEFINITION. Let M be a left A module. A *tertiary decomposition* of a submodule L in M is obtained by writing L as a finite intersection $L = M_1 \cap \ldots \cap M_n$ of tertiary modules M_i of M, such that

(i) the decomposition is irredundant,

(ii) $\mathrm{ass}(M/M_i) \neq \mathrm{ass}(M/M_j)$ for $i \neq j$.

(13.8) THEOREM (Tertiary Decomposition Theorem).

Every submodule L of a left-noetherian A module M has a tertiary decomposition. If $L = M_1 \cap \ldots \cap M_n = N_1 \cap \ldots N_n$ are two tertiary decompositions of L in M, the $m = n$, and

$$\{\mathrm{ass}(M/M_i)\} = \{\mathrm{ass}(M/N_j)\} = \mathrm{Ass}(M/L).$$

PROOF. See [St] p. 160-163. $\triangle$

We now give a structure theorem for the cotertiary left-A-modules.

(13.9) THEOREM (Characterization Theorem for Cotertiary Left Modules).

Let $P = A_{(H,\,\mathrm{eq}(\overline{p_H}))}$ be a prime ideal of A, and let M be a left-A-module. Then M is P-cotertiary if and only if the following two conditions are satisfied:

(1) M is H-cospecial.

(2) $\mathrm{Ass}_{\overline{A(H)}}\Big(\big(\mathrm{chsoc}(M)\big)(H)\Big) = \{{}^g\overline{p_H} \mid g \in W_GH\}$.

PROOF. Assume that M is P-cotertiary. From proposition (13.2) *(3)* it follows that $\mathrm{Char}(M) = [H]$. From theorem (12.10), it follows that M is H-cospecial. In particular, $\mathrm{chsoc}(M)$ is H-characteristic and essential in M. Now let $\overline{q_H}$ be an associated ideal of the left-$\overline{A(H)}$-module $\big(\mathrm{chsoc}(M)\big)(H)$. From theorem (A1.29), it follows that there exists an W_GH-invariant submodule N_H of $\big(\mathrm{chsoc}(M)\big)(H)$ such that $\mathrm{eq}(\overline{q_H}) = \mathrm{Ann}_{\overline{A(H)}}(N'_H)$ for all non-zero W_GH-invariant submodules N'_H of N_H. Let

$$N = A\langle N_H\rangle = \sum_{m\in N_H} A\langle m\rangle.$$

From formula (6.19), it follows that $Q = A_{(H,\,\mathrm{eq}(\overline{q_H}))} = \mathrm{Ann}_A(N')$, for all non-zero submodules N' of N. Since M is P-cotertiary it follows that $Q = P$, hence *(2)*.

Conversely, assume that M satisfies the conditions *(1)* and *(2)*. Let $Q = A_{(H,\,\mathrm{eq}(\overline{q_H}))} \in \mathrm{Ass}_A(M)$, and let N be a submodule of M such that $\mathrm{Ann}_A(N) = \mathrm{Ann}_A(N') = Q$ for all non-zero submodules N' of N. Since $N \cap \mathrm{chsoc}(M) \neq 0$, it follows that we may assume that N is H-characteristic. Let N'_H be an arbitrary non-zero W_GH-invariant submodule of $N(H)$, and again let

$$N' = A\langle N'_H\rangle = \sum_{m\in N'_H} A\langle m\rangle.$$

Notice that $N' \subseteq N$, and

$$A_{(H,\,\mathrm{Ann}_{\overline{A(H)}}\,N'_H)} = \mathrm{Ann}_A(N') = Q = \mathrm{Ann}_A(N) = A_{(H,\,\mathrm{Ann}_{\overline{A(H)}}\,N(H))}.$$

This shows that $\mathrm{eq}(\overline{q_H}) = \mathrm{Ann}_{\overline{A(H)}}(N'_H)$, for all non-zero W_GH-invariant submodules of $N(H)$. From theorem (A1.29), it follows that $\overline{q_H} \in \mathrm{Ass}_{\overline{A(H)}}\big(\mathrm{chsoc}(M))(H)\big)$. Since M satisfies *(2)*, it follows that $\mathrm{eq}(\overline{p_H}) = \mathrm{eq}(\overline{q_H})$, hence $P = Q$. Therefore $\mathrm{Ass}_A(M) = \{P\}$. $\triangle$

As a corollary to theorems (12.21) and (13.8), we conclude that cotertiary modules satisfy the following induction theorem:

(13.10) THEOREM.

Let M be a left A module such that $\mathrm{Ass}_A(M) = \{P\}$. *Then,*

(1) If A/P has characteristic zero, then M is H-characteristic and M satisfies the induction theorem (6.2.5).

(2) If A/P has characteristic $p > 0$, and M is left noetherian then M satisfies the induction theorem (6.2.4) *(2).*

We now give an example of a cotertiary Mackey functor.

(13.11) EXAMPLE. Let G be a p group for some prime number p. For every $H \in S(G)$, let $n_H = \log_p|H|$. Let $M(H) = \mathbf{Z}_{p^{n_H}}$. Then M becomes a Mackey functor with the conjugation being the identity map of $M(H)$, the restriction map r_K^H being the canonical projection of $\mathbf{Z}_{p^{n_H}} \longrightarrow \mathbf{Z}_{p^{n_K}}$, and the transfer map t_K^H being the multiplication with the index $|H/K|$. One can easily show that t_K^H is well defined and that M is indeed a Mackey functor. It is easy to check that M is G-cospecial, hence M is a $B_{(G,\ p\mathbf{Z})}$ cotertiary module (as a module over B).

We now introduce the notion of a primary submodule of a left A module.

(13.12) DEFINITION. Let P be a prime ideal of A. A left-A-module M is called *P-coprimary* if M is P-cotertiary and $P^n \subseteq \mathrm{Ann}_A(M)$ for some $n \geq 1$. We refer to such a module as P-coprimary. A submodule N of a left-A-module M is called *P-primary* if M/N is P-coprimary.

(13.13) THEOREM (Characterization Theorem for Coprimary Modules and Primary Submodules).

Let $P = A_{(H,\ \mathrm{eq}(\overline{p_H}))}$ be a prime ideal of A.

(1) Let M be a left-A-module. Then M is P-coprimary if and only if M is P-cotertiary, H-characteristic, and $\mathrm{Ann}_{\overline{A(H)}}\ \overline{M(H)} \supseteq \mathrm{eq}(\overline{p_H})^n$ *for some $n \geq 1$.*

(2) Again let M be a left-A-module. Then all the P-primary submodules N of M are of the form

$$N = M_{(H,\ \overline{N(H)})}$$

where $\overline{N(H)}$ is an W_GH-invariant submodule of $\overline{M(H)}$ such that

$$\mathrm{Ass}_{\overline{A(H)}}(\overline{M(H)}/\overline{N(H)}) = \{{}^g\overline{p_H} \mid g \in W_GH\},$$

and

$$\left(\mathrm{eq}(\overline{p_H})\right)^n \subseteq \mathrm{Ann}_{\overline{A(H)}}(\overline{M(H)}/\overline{N(H)}) \qquad \text{for some } n \geq 1.$$

PROOF. *(1)* Assume that M is P-coprimary. Then M is also P-cotertiary. Moreover, notice that $P(K) = A(K)$ whenever $[H] \not\leq [K]$. From corollary (3.1.4), it follows that $(P^n)(K) = A(K)$ whenever $[H] \not\leq [K]$ and $n \geq 1$. Since $\mathrm{Ann}_A(M)$ contains P^n for some $n \geq 1$, it follows that $M(K) = 0$, for $K < H$. Since $M \longrightarrow M_{G/H}$ is injective (because M is H-cospecial), it follows that M is H-characteristic. Since M is H-characteristic it follows that $\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{Ann}_{\overline{A(H)}}\overline{M_H})}$. Since $(P^n)(K) = A(K)$ whenever $K < H$, it follows that $\overline{(P^n)(H)} = \overline{P(H)}^n = (\mathrm{eq}(\overline{p_H}))^n$. Since $P^n \subseteq \mathrm{Ann}_A(M)$, for some $n \geq 1$, and both these ideals are $A(K)$, for $K < H$, it follows easily that

$$\mathrm{eq}(\overline{p_H})^n = \overline{P(H)}^n = \overline{(P^n)(H)} \subseteq \overline{(\mathrm{Ann}_A M)(H)} = \mathrm{Ann}_{\overline{A(H)}}\overline{M(H)}.$$

Conversely, assume that M satisfies the three conditions from *(1)*. It is assumed that M is P-cotertiary. Moreover, since M is H-characteristic, it follows that $\mathrm{Ann}_A(M) = A_{(H,\ \mathrm{Ann}_{\overline{A(H)}}\overline{M(H)})}$. Now since $(\mathrm{eq}(\overline{p_H}))^n \subseteq \mathrm{Ann}_{\overline{A(H)}}\overline{M(H)}$, it follows from the previous argument that $(P^n)(H) \subseteq (\mathrm{Ann}_A(M))(H)$. Since $\mathrm{Ann}_A(M)$ is the largest ideal I whose value at H is $(\mathrm{Ann}_A(M))(H)$, it follows immediately that $P^n \subseteq \mathrm{Ann}_A(M)$.

(2) Follows immediately from *(1)*. $\triangle$

We now investigate the coprimary modules when A is a commutative Green functor.

(13.14) THEOREM.

Let A be a commutative Green functor. A noetherian A-module M is coprimary if and only if, whenever $0 \neq m \in M$ and $a \in A$ are such that $a \times m = 0$, then $[a]^n \in \mathrm{Ann}_A(M)$, for some $n \geq 1$. In this case, $\mathrm{ass}_A(M) = \sqrt{\mathrm{Ann}_A(M)}$.

PROOF. Assume that M is coprimary, and let $a \in A$ and $m \in M$ such that m is non-zero and $a \times m = 0$. It follows that $a \in \mathrm{Ann}_A(A\langle m\rangle)$. Since M is P-cotertiary it follows that $\mathrm{Ann}_A(A\langle m\rangle) \subseteq P = \mathrm{ass}_A(M)$. In particular $a \in P$. Since M is coprimary, it follows that $[a]^n \in P^n \subseteq \mathrm{Ann}_A(M)$, for some $n \geq 1$.

Conversely, assume that M is an A-module which satisfies the property that whenever $0 \neq m \in M$ and $a \in A$ are such that $a \times m = 0$, then $[a]^n \in \mathrm{Ann}_A(M)$, for some $n \geq 1$. We check that M satisfies the condition *(1)* from theorem (13.13).

We show that M is cotertiary. Let $Q \in \mathrm{Ass}_A(M)$. There exists $N \subseteq M$, such that N is non-zero and $\mathrm{Ann}_A(N') = Q$, whenever N' is a non-zero submodule of N. In particular, for every $a \in Q$ and every $n \in N$, $a \times n = 0$. We conclude that $[a]^{k_a} \in \mathrm{Ann}_A(M)$, for some $k_a \geq 1$ depending on a. Assume now that P is some other prime ideal in $\mathrm{Ass}_A(M)$. Then since $\mathrm{Ann}_A(M) \subseteq P$, it follows that $[a]^{k_a} \subseteq P$ for all $a \in Q$. Since P is prime, we conclude that $Q \subseteq P$. Since in fact Q and P were chosen arbitrarily in $\mathrm{Ass}_A(M)$, it follows that M is cotertiary. From now on we assume that $Q = \mathrm{ass}_A(M)$.

We now prove that M is characteristic. Assume that $Q = A_{(H,\,\mathrm{eq}(\overline{q_H}))}$. We show that $M(K) = 0$, for $K < H$. Indeed, if $M(K) \neq 0$, for some $K < H$, then let $m \in (\mathrm{chsoc}(M))(H)$ be a non-zero element. It is easy to see that $1_{A(K)} \times m = 0$. We conclude that $[1_{A(K)}]^n = 1_{A((G/K)^n)} \in (\mathrm{Ann}_A(M))((G/K)^n)$, for some $n \geq 1$. However, since the set $(G/K)^n$ has orbits of the form G/K, it follows that $1_{A(K)} \in (\mathrm{Ann}_A(M))(K)$; hence $M(K) = 0$. This argument shows that M is H-bounded. Since M is Q-cotertiary (in particular, H-cospecial), it follows that the map $M \longrightarrow M_{G/H}$ is injective. We conclude that M is H-characteristic.

Finally we show that $\mathrm{Ann}_{\overline{A(H)}}M(H) \supseteq \left(\mathrm{eq}(\overline{q_H})\right)^n$, for some $n \geq 1$. Since $\overline{A(H)}$ is noetherian, it follows that $\overline{Q(H)}$ is finitely generated. Now we know that, for all $a \in Q$, $[a]^{k_a} \in \mathrm{Ann}_A(M)$ for some $k_a \geq 1$. From proposition (5.2), it follows that, whenever $a \in Q(H)$, there exists $s_a \geq 1$ such that $a^{s_a} \in \mathrm{Ann}_{A(H)}(M(H))$. Since $M(H) = \overline{M(H)}$ is an $\overline{A(H)}$-module, it follows that, for every $\overline{a} \in \overline{Q(H)}$, $\overline{a}^{s_a} \in \mathrm{Ann}_{\overline{A(H)}}\overline{M(H)}$. Since $\overline{Q(H)} = \mathrm{eq}(\overline{q_H})$ is finitely generated, it follows that $\overline{Q(H)}^s \subseteq \mathrm{Ann}_{\overline{A(H)}}\overline{M(H)}$, for some $s \geq 1$. Hence M is primary. Finally, the fact that $\mathrm{ass}_A(M) = \sqrt{\mathrm{Ann}_A(M)}$ follows because $\mathrm{ass}_A(M)^n \subseteq \mathrm{Ann}_A(M) \subseteq \mathrm{ass}_A(M)$, for some $n \geq 1$. $\triangle$

(13.15) EXAMPLE. Let R be a noetherian ring and let B be the Burnside ring Green functor. We know that $\overline{B(H)} = R$ and W_GH acts trivially on R. Let $P = B_{(H,\,p)}$ be an arbitrary prime ideal of B. Then, acording to theorem (13.13), the only P-primary ideals in B are exactly the ideals $B_{(H,\,p')}$, where p' is an ideal of R which is p-primary in R. In particular, when $R = \mathbf{Z}$, the only $B_{(H,\,0)}$-primary ideal in B is itself. If $p > 0$, then the only $B_{(H,\,p\mathbf{Z})}$-primary ideals in B are exactly the ideals $B_{(H,\,p^n\mathbf{Z})}$, for $n \geq 1$. This result was obtained earlier by Lewis.

We investigate the circumstances under which the primary decomposition theorem holds.

(13.16) THEOREM.

Let A be a commutative Green functor, and L be a submodule of a noetherian A module M. Then L can be written as a finite intersection $L = N_1 \cap \ldots \cap N_k$ of primary submodules in M if and only if the map $\beta : M/L \longrightarrow T(M/L)$ is injective.

PROOF. Assume that L can be written as a finite interesection of primary submodules of M. Each of this submodules is in particular cocharacteristic in M. From theorem (6.1.22), it follows that the map $\beta : M/L \longrightarrow T(M/L)$ is injective.

Conversely, if the map $\beta : M/L \longrightarrow T(M/L)$ is injective, it follows from theorem (6.1.22), that L can be written as an intersection of finitely many cocharacteristic submodules. We show that every cocharacteristic submodule of M can be written as a finite intersection of primary submodules in M. Assume that $N = M_{(H,\overline{N(H)})}$ is H-characteristic. Since $\overline{M(H)}$ is noetherian, it follows, from theorem (A1.37), that

$$\overline{N(H)} = \bigcap_{i=1}^{j} \mathrm{eq}(\overline{N_{i,H}})$$

where $N_{i,H}$ is a primary submodule of $\overline{M(H)}$, for all $1 \leq i \leq j$. Since the operator $M_{(H,\,-)}$ commutes with the intersections it follows that

$$N = M_{(H,\overline{N(H)})} = \bigcap_{i=1}^{j} M_{(H,\,\mathrm{eq}(\overline{N_{i,H}}))},$$

and, according to theorem (13.13), each one of the submodules $M_{(H,\,\mathrm{eq}(\overline{N_{i,H}}))}$ is primary in M. $\triangle$

From example (6.1.23) and theorem (13.16), we conclude that the ideal 0 is not an intersection of primary ideals of the Burnside ring B for G (over $\mathbf{Z}_p$) if $p > 0$ is a prime number which divides the order of G.

(13.17) EXAMPLE. Let G be a group such that $|G|$ is invertible in $A(G)$ (or R). Then every H-cospecial module is H-characteristic. Indeed, if M is such a module then

$$M = TM = \bigoplus_{K \in [G \backslash \mathcal{P}(M)]} J_{G/K}(\overline{M(K)}),$$

and each one of the submodules $J_{G/K}(\overline{M(K)})$ is obviously K-characteristic. Moreover, the only K-characteristic submodules of M are obtained for $K \in \mathcal{P}(M)$, and all such submodules need to be contained in $J_{G/K}(\overline{M(K)})$. Since M must have an essential H-characteristic submodule, it follows that $\mathcal{P}(M) = \mathrm{Char}(M) = [H]$, and $M = J_{G/H}(M(H))$. Moreover, if A is commutative and noetherian, then every noetherian tertiary module is, of course, primary.

14. Completion.

In this chapter, we investigate the primordial subgroups of the completion of a Mackey functor with respect to a descending chain of subfunctors. As an application, we show that if A is a Green functor and I is a functorial ideal then $D(A/I) = D(\widehat{A}\,)$. Our approach to completion is functorial; that is

$$\widehat{A} = \varprojlim_{n} A/I^n$$

where I^n is the n'th functorial power of I. In various papers (see [A], [MM]) the authors have outlined a different approach to the completion of A in the I-adic topology. The disadvantage of their procedure is that it does not lead to a Green functor unless some technical conditions are satisfied. We refer to this approach as the *ad-hoc* completion. We show that whenever the ad-hoc completion is a Green functor in a canonical way then it is isomorphic to $\widehat{A}$. This result suggests that one should always think of completion in the functorial sense.We end with a couple of examples.

Let us suppose that $(M_n,\ \phi_{n+1})_{n\geq 1}$ is an inverse system of Mackey functors for G (over R), that is

$$\phi_{n+1} : M_{n+1} \longrightarrow M_n, \qquad \text{for all } n \geq 1,$$

is a morphism of Mackey functors. We make the assumption that ϕ_n is onto, for all $n \geq 2$. For every $n \geq 2$, let $N_n = \operatorname{Ker} \phi_n$. Notice that we have an exact sequence

$$0 \longrightarrow N_{n+1} \longrightarrow M_{n+1} \xrightarrow{\phi_{n+1}} M_n \longrightarrow 0. \tag{14.1}$$

Now, if $H \in S(G)$, the system $\big(M_n(H),\ \phi_{n+1}(H)\big)_{n\geq 1}$ is an inverse system of R modules. Let

$$M(H) = \varprojlim_{n} M_n(H), \qquad \text{for all } H \in S(G). \tag{14.2}$$

Since ϕ_n is functorial, for all $n \geq 2$, it follows that the family $\big(M(H)\big)_{H\in S(G)}$ is, in fact, a Mackey functor. We denote it by

$$M = \varprojlim_{n} M_n, \tag{14.3}$$

and refer to it as *the inverse limit of the system* $(M_n)_{n\geq 1}$. Notice that, if A is a commutative Green functor, the M_n are A modules, and the ϕ_{n+1} are morphisms of A modules, for all $n \geq 1$, then

$$M = \varprojlim_n M_n$$

is an A module as well.

The main result of this chapter is the following theorem.

(14.1) THEOREM.

Assume that $\mathcal{P} \subseteq S(G)$ *is such that* $\mathcal{P}(M_n) \cap \mathcal{P}(N_{n+1}) \subseteq \mathcal{P}$ *for all* $n \geq 1$. *Then*

$$\mathcal{P}(\varprojlim_n M_n) \subseteq \mathcal{P}.$$

PROOF. Fix $H \in S(G)$. Denote by $\mathcal{P}_H = \mathcal{P} \cap S(H)$. We view

$$M(H) = (\varprojlim_n M_n)(H) = \varprojlim_n M_n(H)$$

as an R-submodule of

$$\prod_{n=1}^{\infty} M_n(H).$$

With this convention, we identify $M(H)$ with the set

$$\Big\{\big(x_n(H)\big)_{n\geq 1} \;\Big|\; x_n(H) \in M_n(H) \text{ such that } \phi_{n+1}(x_{n+1}(H)) = x_n(H), \text{ for all } n \geq 1\Big\}.$$

Similarly, we view

$$\bigoplus_{K\in\mathcal{P}_H} M(K) = \bigoplus_{K\in\mathcal{P}_H} (\varprojlim_n M_n)(K) = \varprojlim_n \Big(\bigoplus_{K\in\mathcal{P}_H} M_n(K)\Big),$$

as an R-submodule of

$$\prod_{n=1}^{\infty}\Big(\bigoplus_{K\in\mathcal{P}_H} M_n(K)\Big).$$

With this convention, we identify $\bigoplus_{K\in\mathcal{P}_H} M(K)$ with the set

$$\Big\{\big(\sum_{K\in\mathcal{P}_H} x_n(K)\big)_{n\geq 1} \;\Big|\; x_n(K) \in M_n(K) \text{ and } \phi_{n+1}(x_{n+1}(K)) = x_n(K),\ K \in \mathcal{P}_H,\ n \geq 1\Big\}.$$

We show that, for all $x(H) \in M(H)$, there exist elements $y(K)$ of $M(K)$, for $K \in \mathcal{P}_H$, such that

$$x(H) = \sum_{K\in\mathcal{P}_H} t_K^H\, y(K).$$

Assume that $x(H) = \big(x_n(H)\big)_{n\geq 1}$. Suppose that, for some $m \geq 1$, we have constructed

$$\Big(y_j(K)\Big)_{\substack{1\leq j\leq m \\ K\in\mathcal{P}_H}}$$

such that $y_j(K) \in M_j(K)$,

$$x_j(H) = \sum_{P\in\mathcal{P}_H} t_K^H\, y_j(K), \qquad \text{for } 1 \leq j \leq m,$$

and

$$\phi_{j+1}\big(y_{j+1}(K)\big) = y_j(K), \qquad \text{for } 1 \leq j < m.$$

We show how to construct elements $y_{m+1}(K)$ of $M_{n+1}(K)$, for $K \in \mathcal{P}_H$, such that

$$x_{m+1}(H) = \sum_{K\in\mathcal{P}_H} t_K^H\, y_{m+1}(K), \tag{14.4}$$

and

$$\phi_{m+1}\big(y_{m+1}(K)\big) = y_m(K), \qquad \text{for } K \in \mathcal{P}_H. \tag{14.5}$$

First of all, for $K \in \mathcal{P}_H$, we choose $y'_{m+1}(K)$ such that $\phi_{m+1}\big(y'_{m+1}(K)\big) = y_m(K)$. Let

$$x'_{m+1}(H) = \sum_{K\in\mathcal{P}_H} t_K^H\, y'_{m+1}(K).$$

Notice that

$$\phi_{m+1}\big(x_{m+1}(H) - x'_{m+1}(H)\big) = x_m(H) - \sum_{K\in\mathcal{P}_H} \phi_{m+1}\big(t_K^H\, y'_{m+1}(K)\big) =$$

$$x_m(H) - \sum_{K\in\mathcal{P}_H} t_K^H\, \Big(\phi_{m+1}\big(y'_{m+1}(K)\big)\Big) = x_m(H) - \sum_{K\in\mathcal{P}_H} t_K^H\, y_m(K) = 0.$$

It follows that $x_{m+1}(H) - x'_{m+1}(H) \in \big(\operatorname{Ker}\phi_m\big)(H) = N_{m+1}(H)$. Since $\mathcal{P}(N_{m+1}) \subseteq \mathcal{P}$, it follows that we can find elements $y^*_{m+1}(K)$ of $N_{m+1}(K)$, for $K \in \mathcal{P}_H$, such that

$$x_{m+1}(H) - x'_{m+1}(H) = \sum_{K\in\mathcal{P}_H} t_K^H\, y^*_{m+1}(K).$$

Hence

$$x_{m+1}(H) = x'_{m+1}(H) + \sum_{K\in\mathcal{P}_H} t_K^H\, y^*_{m+1}(K) = \sum_{K\in\mathcal{P}_H} t_K^H\, y'_{m+1}(K) + \sum_{K\in\mathcal{P}_H} t_K^H\, y^*_{m+1}(K) =$$

$$\sum_{K\in\mathcal{P}_H} t_K^H\big(y'_{m+1}(K) + y^*_{m+1}(K)\big). \tag{14.6}$$

Let $y_{m+1}(K) = y'_{m+1}(K) + y^*_{m+1}(K)$, for $K \in \mathcal{P}_H$. We claim that the elements $y_{m+1}(K) \in M_{m+1}(K)$ satisfy equations (14.4) and (14.5). Notice that (14.5) follows from (14.6). For (14.4), notice that

$$\phi_{m+1}\big(y_{m+1}(K)\big) = \phi_{m+1}\big(y'_{m+1}(K) + y^*_{m+1}(K)\big) =$$

$$\phi_{m+1}\big(y'_{m+1}(K)\big) + \phi_{m+1}\big(y^*_{m+1}(K)\big) = y_m(K),$$

because $\phi_{m+1}\big(y'_{m+1}(K)\big) = y_m(K)$, and $y^*_{m+1}(K) \in N_{m+1} \in \mathrm{Ker}\ \big(\phi_{m+1}(K)\big)$. $\triangle$

Let now M be a Mackey functor. Suppose that

$$M_1 \supset M_2 \supset \ldots \supset M_n \supset \ldots \tag{14.7}$$

is a descending chain of sub-functors of M. For $n \geq 1$, let

$$\phi_{n+1} : \frac{M}{M_{n+1}} \longrightarrow \frac{M}{M_n}$$

be the canonical projection. Then $\big(M/M_n,\ \phi_{n+1}\big)_{n\geq 1}$ is an inverse system of Mackey functors. The inverse limit of this system is denoted $\widehat{M}$ and is called *the completion of M with respect to the descending chain* (14.7). Notice that the maps ϕ_n are surjective, for $n \geq 2$, and that

$$\mathrm{Ker}\ \phi_{n+1} = \frac{M_n}{M_{n+1}}, \qquad \text{for } n \geq 1.$$

Notice also that, if A is a Green functor, M is an A module, and the M_n are submodules of A for all $n \geq 1$, then $\widehat{M}$ is an A module.

The following corollary is an obvious consequence of theorem (14.1).

(14.2) COROLLARY.

Let $\mathcal{P} \subset S(G)$. Assume that $\mathcal{P}(M/M_n) \cup \mathcal{P}(M_n/M_{n+1}) \subseteq \mathcal{P}$, for all $n \geq 1$. Then $\mathcal{P}(\widehat{M}\ ) \subseteq \mathcal{P}$.

Let A be a Green functor, and let I be an ideal of A. Let $M_1 = I$, and $M_n = I^n$, for $n > 1$. The completion of A with respect to the descending sequence

$$I \supset I^2 \supset \ldots \supset I^n \supset \ldots \tag{14.8}$$

is called *the completion of A with respect to I*, or *the completion of A with respect to the I-adic topology* and it is denoted $\widehat{A}$. We have the following result.

(14.3) THEOREM.

$\mathcal{P}(\widehat{A}\,) = \mathcal{P}(A/I)$. *In particular,* $\mathcal{D}(\widehat{A}\,) = \mathcal{D}(A/I)$.

PROOF. It is clear that A/I is an epimorphic image of $\widehat{A}$. This shows that $\mathcal{P}(A/I) \subseteq \mathcal{P}(\widehat{A}\,)$. For the reverse containment, we apply theorem (4.1.15) *(2)* to conclude that $\mathcal{P}(A/I^n) \subseteq \mathcal{P}(A/I)$ for all $n \geq 1$. Moreover, since I^n/I^{n+1} is an A/I module, it follows that $\mathcal{P}(I^n/I^{n+1}) \subseteq \mathcal{P}(A/I)$. From corollary (14.2), it follows that $\mathcal{P}(\widehat{A}\,) \subseteq \mathcal{P}(A/I)$. $\triangle$

We now outline the ad-hoc approach to the completion of a Green functor A in a functorial ideal I. Let $H \in S(G)$. We give $A(H)$ the $I(H)$-adic topology. Denote by $\widehat{A(H)}$ the completion of the ring $A(H)$ with respect to the $I(H)$-adic topology. It is clear that if $K < H$, then the map $r_H^K : A(H) \longrightarrow A(K)$ is continuous; hence it induces a map $\widehat{r_K^H} : \widehat{A(H)} \longrightarrow \widehat{A(K)}$. Similarly, if $g \in G$, the map $c_g : A(H) \longrightarrow A({}^gH)$ is continuous; hence it induces a map $\widehat{c_g} : \widehat{A(H)} \longrightarrow \widehat{A({}^gH)}$. We conclude that, if the maps $t_K^H : A(K) \longrightarrow A(H)$ are continuous, then the collection $\widehat{A(H)}_{H \in S(G)}$ is a Green functor in a canonical way. The connection between $(\widehat{A(H)})_{H \in S(G)}$ and $\widehat{A}$ is given by the following theorem.

(14.4) THEOREM.

Let I be an ideal of A. For each $H \in S(G)$, assume that $A(H)$ is given the $I(H)$-adic topology. If $t_K^H : A(K) \longrightarrow A(H)$ is continuous, for all $K < H \in S(G)$, then the resulting Green functor $(\widehat{A(H)})_{H \in S(G)}$ is canonically isomorphic with $\widehat{A}$.

PROOF. It is enough to show that if $H \in S(G)$, then the two filtrations

$$I(H) \supset I(H)^2 \supset ... \supset I(H)^n \supset ... \tag{14.9}$$

and

$$I(H) \supset (I^2)(H) \supset ... \supset (I^n)(H) \supset ... \tag{14.10}$$

give equivalent topologies on $A(H)$. Notice that $I(H)^m \subseteq (I^m)(H)$ for all $m \geq 1$. Since the maps $t_K^H : A(K) \longrightarrow A(H)$ are continuous for all $K < H$, it follows that, for every $m \geq 1$ and $K < H$, there exists $n = n_{m,K}$, such that $t_K^H(I(K)^n) \subseteq I(H)^m$. Let $n_m = \max\,(m,\ n_{m,K} \mid K < H)$. Then $t_K^H(I(K)^{n_m}) \subseteq I(H)^m$ for all $K < H$. From formula (3.16), it follows that

$$(I^{n_m})(H) \subseteq I(H)^m.$$

Since m was arbitrary, it follows that the topology given by filtration (14.10), is equivalent to the $I(H)$-adic topology of $A(H)$. $\triangle$

The above result suggests that one should always think of completion in the functorial sense, i.e. completing the Green functor A in the I-adic topology given by the ideals I^n, instead of thinking of the completion of $A(H)$ in the $I(H)$-adic topology for all $H \in S(G)$. The reason is that the first completion will always exist, whereas the second one does not exist unless $t_K^H : A(K) \longrightarrow A(H)$ is continuous for all $K < H$, in which case it coincides with the first one.

The following example shows that $t_K^H : A(K) \longrightarrow A(H)$ is not always continuous.

(14.5) EXAMPLE. Let B be the Burnside ring Green functor for G over $\mathbf{Z}$. Assume that p is a prime number dividing $|G|$, and let H be a subgroup of order p in G. Let $I = B_{(H,0)}$. Since $I(1) = B(1) = \mathbf{Z}$, it follows that $I(1)^n = B(1) = \mathbf{Z}$. Hence, from formula (3.16), it follows that for all $n \geq 1$, $(I^n)(H) \supseteq t_1^H(I(1)^n) = t_1^H(B(1)) = I(H)$. Since $(I^n)(H) \subseteq I(H)$ for all $n \geq 1$, it follows that $(I^n)(H) = I(H)$ for $n \geq 1$. In particular, for this ideal, filtration (14.10) is stationary. However, it is easy to see that the filtration $I(H)^n$ is strictly decreasing. Since these two filtrations do not give equivalent topologies on $B(H)$, it follows that $\left(\widehat{B(H)}\right)_{H \in S(G)}$ is not a Green functor.

If M is a left-A-module then the completion of M at I is the completion of M with respect to the descending chain

$$IM \supseteq I^2M \supseteq \ldots \supseteq I^nM \supseteq \ldots$$

(14.6) PROPOSITION.

Let $\widehat{A}$ and $\widehat{M}$ be the completions of A and M at I. Then $\widehat{M}$ is a left-$\widehat{A}$-module. In particular, $\widehat{M}$ is projective relative to $D(A/I)$.

We end with a couple examples for theorem (14.3).

(14.7) EXAMPLE. Let $R_{\mathbf{C}}$ be the character ring Green functor and let $I = (R_{\mathbf{C}})_{(1,0)}$ be the functorial ideal of all zero-dimensional characters. We give $R_{\mathbf{C}}(H)$, the $I(H)$-adic topology. As was proven by Atiyah, the transfer maps $t_K^H : R_{\mathbf{C}}(K) \longrightarrow R_{\mathbf{C}}(H)$ are continuous, hence $\left((\widehat{R_{\mathbf{C}}(H)}\right) = \widehat{R_{\mathbf{C}}}$. From theorem (14.3) it follows that $\mathcal{P}(\widehat{R_{\mathbf{C}}}\ ) = \mathcal{P}(R_{\mathbf{C}}/I)$. Since $R_{\mathbf{C}}/I$ is cohomological, from proposition (4.1.8) it follows that $R_{\mathbf{C}}/I$ is $\coprod_p G/S_p$-projective, where S_p is a p-Sylow subgroup of G. Hence $\widehat{R_{\mathbf{C}}}$ is $\coprod_p G/S_p$-projective as well. Actually Atiyah has proved more, namely that $\widehat{R_{\mathbf{C}}}$ is canonically isomorphic with $\mathcal{K}^*(B_G)$, the topological K-theory of the classifying space B_G. Since $\mathcal{K}^*(B_G)$ is a cohomological Green functor, it follows again (without the help of (14.3)) that the only primordial subgroups of $\mathcal{K}^*(B_G)$ are p-groups. See [A] for more details.

(14.8) EXAMPLE. Let B be the Burnside ring Green functor for G (over R), and let $I = B_{(1,0)}$ be the augmentation ideal of B. It is known that B/I is a cohomological Green functor. It follows from proposition (4.1.8) that all the primordial subgroups of B/I are the p-groups, such that p is a prime number which is not invertible in R. Let Π be the set of these prime numbers. Then B/I is $\coprod_{p\in\Pi} G/S_p$-projective. We now give $B(H)$ the $I(H)$-adic topology. May and McClure have shown that the maps $t_K^H : B(K) \longrightarrow B(H)$ are continuous, for all $K < H \in S(G)$. In particular, $\widehat{B(H)}_{H\in S(G)} = \widehat{B}$. From theorem (14.3), it follows that $\widehat{B}$ is $\coprod_{p\in\Pi} G/S_p$-projective. In particular, $\widehat{B(H)}_{H\in S(G)}$ is $\coprod_{p\in\Pi} G/S_p$-projective as well. In this form the result appears in [MM].

A. Appendix.

This appendix has two parts. In A1 we investigate various properties of modules over rings with group actions. In A2 we state miscellaneous results from classical algebra.

A1. RINGS AND MODULES WITH GROUP ACTIONS.

Throughout this section R is a commutative ring with unity 1_R, S is an associative ring with unity 1_S which is a unitary R-algebra, and G is a finite group acting on S (not necessarily effectively) by R-linear unitary ring automorphisms. If $s \in S$ and $g \in G$, then the action of g on s is denoted with $c_g(s)$. In this chapter we introduce the notion of a *G-equivariant* left-S-module. We study the annihilators of G-equivariant simple left-S-modules. We define the equivariant Jacobson radical of S as being the intersection of annihilators of all G-equivariant left-S-modules. We show that it coincides with the classical Jacobson radical of S. We also investigate the analogs of prime and maximal ideals in this setting and we prove structure theorems for them. For left-noetherian rings S we investigate the analogs of primary and tertiary G-invariant submodules of a G-equivariant left-S-module and we prove structure theorems for them.

(A1.1) DEFINITION (The Twisted Group Algebra). The *twisted group algebra of G with coefficients in S*, denoted by $T^{\frown}S[G]$ is an S-algebra defined as follows:

-As a left-S-module, $T^{\frown}S[G]$ is the free S-module on the set G, i.e. every element of $x \in T^{\frown}S[G]$ can be written in a unique way as

$$x = \sum_{g \in G} s_g \cdot g, \qquad \text{for some } s_g \in S \text{ and } g \in G. \tag{A1.1}$$

-If $g_1,\ g_2 \in G$, and $s_1,\ s_2 \in S$, then the product of $s_1 \cdot g_1 \in T^{\frown}S[G]$ with $s_2 \cdot g_2 \in T^{\frown}S[G]$ is taken to be

$$(s_1 \cdot g_1)(s_2 \cdot g_2) = \big(s_1 c_{g_1}(s_2)\big) \cdot (g_1 g_2). \tag{A1.2}$$

The product of two arbitrary elements of $T^{\frown}S[G]$ is obtained by extending rule $(A1.2)$ to $T^{\frown}S[G]$ by linearity.

One can easily check that $T^{\sim}S[G]$ is an associative ring with unity $1_S \cdot e$, where e is the identity element of the group G. Moreover, $T^{\sim}S[G]$ is both a left and right S-module, and an S-S-bimodule. The ring S can be identified with a subring of $T^{\sim}S[G]$ via the injection

$$S \longrightarrow T^{\sim}S[G], \qquad s \longmapsto s \cdot e.$$

With this identification, $T^{\sim}S[G]$ becomes an S-algebra. It is clear that $T^{\sim}S[G]$ is also an R-algebra, where the action of $\mathbf{r} \in R$ on $s \cdot g \in T^{\sim}S[G]$ is given by

$$(r,\ s \cdot g) \longrightarrow (r(s \cdot g)) = (rs) \cdot g. \tag{A1.3}$$

The action of $r \in R$ on $T^{\sim}S[G]$ is obtained by extending rule (A1.3) to $T^{\sim}S[G]$ by linearity.

(A1.2) EXAMPLE. If G acts trivially on S, then $T^{\sim}S[G]$ is exactly $S[G]$, the group algebra of G with coefficients in S.

(A1.3) DEFINITION (G-Equivariant Left-S-Modules).

A left-S-module M is called G-*equivariant*, if G acts on M by R-linear automorphisms in such a way that the actions of G on S and respectively M are compatible. By the compatibily condition, we mean that if $g \in G$, $s \in S$ and $m \in M$, and if we denote the action of g on m by $c_g(m0$, then

$$c_g(sm) = c_g(s)c_g(m).$$

One can similarly define the notion of a G-equivariant right-S-module.

(A1.4) EXAMPLE. S is an G-equivariant left (and right) S-module over itself.

(A1.5) EXAMPLE. For $g \in G$, define $C_g \in \mathrm{Aut}_R(T^{\sim}S[G])$by

$$C_g(s \cdot g_1) = c_g(s) \cdot (gg_1), \qquad \text{for all } s \in S, \text{ and } g_1 \in G.$$

One can easily check that the assignment

$$g \in G \longmapsto C_g \in \mathrm{Aut}_R(T^{\sim}S[G])$$

defines an action of G on $T^{\sim}S[G]$ by R-linear unitary ring automorphisms. With this action, $T^{\sim}S[G]$, as a left (right) S-module, becomes a G-equivariant left (right) S-module.

(A1.6) PROPOSITION.

There is a one-to-one correspondence between G-equivariant left-S-modules, and left $T^{\frown}S[G]$-modules.

PROOF. Let M be a G-equivariant left-S-module. Then M becomes a left-$T^{\frown}S[G]$-module as follows; if $s \in S$, $g \in G$, and $m \in M$, let

$$(s \cdot g,\ m) \longmapsto sc_g(m). \tag{A1.4}$$

Rule (A1.4) can be extended by linearity to a pairing

$$T^{\frown}S[G] \times M \longrightarrow M.$$

One can easily check that M is a left-$T^{\frown}S[G]$-module via this pairing.

Conversely, let M be a left-$T^{\frown}S[G]$-module. Since $T^{\frown}S[G]$ is an S-algebra, M is a left-S-module. For $g \in G$, and $m \in M$, let $c_g(m) = (1_S \cdot g)m$. One can easily check that M becomes a G-equivariant left-S-module. Moreover, it is clear that the above two correspondences are inverses. $\triangle$

The above correspondence can be rephrased as follows. Let *G-S-Mod*, be the category whose objects are the G-equivariant left-S-modules. The morphisms in *G-S-Mod* are the morphisms of left-S-modules which commute with the action of G in the obvious sense. Then proposition (A1.6) asserts that there exists an equivalence of categories between *G-S-Mod* and $T^{\frown}S[G]$-*Mod*. Throughout this section, we work with *G-S-Mod* although, due to proposition (A1.6), we might as well work with the category $T^{\frown}S[G]$.

Let $M \in$ *G-S-Mod*. Notice that a submodule N of M is G-equivariant if and only if it is G-invariant. Let N be a submodule of M. For $g \in G$, the submodule $c_g(N)$ is denoted by ${}^gN$. The submodule $\bigcap_{g\in G} {}^gN$ is denoted by $\mathrm{eq}(N)$. Notice that $\mathrm{eq}(N)$ is the largest G-invariant submodule of M contained in N. Let $\langle N\rangle_G = \sum_{g\in G} {}^gN$. Notice that $\langle N\rangle_G$ is the smallest G-invariant submodule of M containing N.

(A1.7) DEFINITION (Simple G-Equivariant Left-S-Modules). A G-equivariant left-S-module M is called ***simple*** if $M \neq 0$ and the only G-invariant submodules of M are zero and M.

Notice that a G-equivariant simple left-S-module is just a simple left-$T^{\frown}S[G]$-module (via the correspondence from proposition (A1.6)).

We show that the intersection of annihilators of all simple G-equivariant left-S-modules is the classical Jacobson radical $Jac(S)$ of S. We need some preliminary results.

(A1.8) DEFINITION. Two ideals I and J of S are called *coprime* if $I + J = S$.

(A1.9) LEMMA.

Let I_1, I_2, ..., I_n and J be ideals of S. If I_i and J are coprime for every $i \in \{1, 2, ..., n\}$, then $\bigcap_{i=1}^{n} I_i$ and J are coprime.

PROOF. See [T4], lemma (55.2), p. 514. Δ

Let m be a maximal ideal of S and let $g \in G$. Notice that ${}^g m$ is a maximal ideal of S as well.

(A1.10) LEMMA.

Let m and n be two maximal ideals of S which are not conjugate of one another by an element of G. Then $\mathrm{eq}(m)$ *and* $\mathrm{eq}(n)$ *are coprime.*

PROOF. Since m and ${}^g n$ are coprime for every $g \in G$, it follows, by lemma (A1.9), that m and $\mathrm{eq}(n)$ are coprime. Since $\mathrm{eq}(n)$ is G-invariant, it follows that $\mathrm{eq}(n)$ and ${}^g m$ are coprime for every $g \in G$. From lemma (A1.9) we conclude that $\mathrm{eq}(n)$ and $\mathrm{eq}(m)$ are coprime. Δ

(A1.11) THEOREM (Annihilators of Simple G-Equivariant Left-S-Modules).

(1) Let m be a maximal ideal of S. Then there exists a simple G-equivariant left-S-module M, such that $\mathrm{Ann}_S(M) = \mathrm{eq}(m)$.

(2) Let M be a simple G-equivariant left-S-module. Then M is finitely generated.

(3) Let M be a simple G-equivariant left-S-module. Then $Jac(S) \subseteq \mathrm{Ann}_S(M)$.

(4) Let S be a commutative ring and let M be a simple G-equivariant S-module. Then $\mathrm{Ann}_S(M) = \mathrm{eq}(m)$ *for some maximal ideal m of S.*

PROOF. *(1)* Notice that $S/\mathrm{eq}(m)$ is a G-equivariant left-S-module. However, $S/\mathrm{eq}(m)$ is a ring as well. From Zorn's lemma, it follows that the ring $S/\mathrm{eq}(m)$ has a maximal G-equivariant left ideal I. Let

$$M = \left(\frac{S/\mathrm{eq}(m)}{I}\right).$$

It is clear that M is a simple G-equivariant left-S-module. Moreover, notice that $\mathrm{eq}(m) \subseteq \mathrm{Ann}_S(M)$. We show that $\mathrm{eq}(m) = \mathrm{Ann}_S(M)$. Indeed, let n be a maximal ideal containing $\mathrm{Ann}_S(M)$. Since $\mathrm{Ann}_S(M)$ is G-invariant it follows that $\mathrm{Ann}_S(M) \subseteq \mathrm{eq}(n)$. Hence

$$\mathrm{eq}(m) \subseteq \mathrm{Ann}_S(M) \subseteq \mathrm{eq}(n). \qquad (A1.5)$$

It follows, by lemma (A1.10), that m and n are conjugate of one another by an element of G. Hence $\mathrm{eq}(m) = \mathrm{eq}(n)$. From formula (A1.5) it follows that $\mathrm{Ann}_S(M) = \mathrm{eq}(m)$.

(2) Choose $x \neq 0$, $x \in M$. Then the submodule $\langle x \rangle_G = \sum_{g \in G} S \cdot c_g(x)$ of M is G-invariant and non-zero. From the G-equivariant simplicity of M we conclude that $M = \langle x \rangle_G$. Hence, M is finitely generated.

(3) Since M is finitely generated, it follows, by Nakayama lemma, that $Jac(S) \cdot M$ is a proper submodule of M. Since $Jac(S)$ is G-invariant, it follows that $Jac(S) \cdot M$ is a proper G-invariant submodule of M. Since M is simple G-equivariant it follows that $Jac(S) \cdot M = 0$.

(4) Since M is finitely generated, it follows, by Zorn's lemma, that M has a maximal submodule N. Since M/N is simple and S is commutative, it follows that $\mathrm{Ann}_S(M/N) = m$ for some maximal ideal m of S. Then $\mathrm{Ann}_S(M/{}^gN) = {}^gm$. Notice that $\mathrm{eq}(N)$ is a proper G-invariant submodule of M. Since M is simple G-equivariant, it follows that $\mathrm{eq}(N) = 0$. Hence

$$\mathrm{Ann}_S(M) = \mathrm{Ann}_S(M/\mathrm{eq}(N)) = \bigcap_{g \in G} \mathrm{Ann}_S(M/{}^gN) = \bigcap_{g \in G} {}^gm = \mathrm{eq}(m). \qquad \triangle$$

(A1.12) DEFINITION (The G-Equivariant Left Jacobson Radical of S). The *G-equivariant left Jacobson radical of S* is defined as the intersection of annihilators of all simple G-equivariant left-S-modules. According to theorem (A1.11) this coincides with the Jacobson radical $Jac(S)$ of the ring S. We conclude that if we define similarly the G-equivariant right Jacobson radical of S as the intersection of the annihilators of all simple G-equivariant right-S-modules, then the two G-equivariant Jacobson radicals are equal and they are also equal to $Jac(S)$.

We now investigate the equivariant chain conditions.

(A1.13) G-EQUIVARIANT CHAIN CONDITIONS.

(1) A G-equivariant left-S-module M is called ***G-equivariantly left-noetherian*** (respectively ***G-equivariantly left-artinian***) if M satisfies the ascending (descending) chain condition on the set of all G-invariant submodules of M.

(2) The ring S is called ***G-equivariantly left-noetherian*** (***G-equivariantly left-artinian***) if S is G-equivariantly left-noetherian (G-equivariantly left-artinian) as a module over itself.

(3) The ring S is called ***G-equivariantly noetherian*** (respectively ***G-equivariantly artinian***) if S satisfies the ascending (descending) chain condition on the set of all G-invariant

ideals of S.

Let S^G be the set of all G-invariant elements of S. Notice that S^G is an R-algebra. For M a G-equivariant left-S-module, let M^G be the set of all G-invariant elements of M. Then M^G is a left-S^G-module. We have the following result due to Fisher.

(A1.14) THEOREM ([F])

(1) A G-equivariant left-S-module M is G-equivariantly left-noetherian (respectively G-equivariantly left-artinian) if and only if M is left-noetherian (respectively left-artinian). The ring S is G-equivariantly noetherian (artinian) if and only if S is noetherian (artinian).

(2) Assume that $|G|$ is invertible in S. Let M be a G-equivariant left-noetherian (left-artinian) S-module. Then M^G is a left-noetherian (left-artinian) S^G-module.

We now investigate the G-maximal and and G-prime ideals of the ring S.

(A1.15) DEFINITION.

(1) An ideal I of S is called *G-maximal* if I is a maximal proper G-invariant ideal of S.

(2) An ideal P of S is called *G-prime* if P is a proper G-invariant ideal of S which satisfies the following property:

-whenever I and J are two G-invariant ideals of S such that $I \cdot J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

(A1.16) DEFINITION. (G-Simple and G-Prime Rings). The ring S is called *G-simple* if the ideal 0 is G-maximal. The ring S is called *G-prime* if the ideal 0 is G-prime. When S is commutative and G-simple, we refer to S as being a *G-field*. When S is commutative and G-prime we refer to S as being a *G-domain*.

(A1.17) THEOREM (The Structure of G-Maximal Ideals of S).

An ideal I of S is G-maximal if and only if $I = \mathrm{eq}(m)$ for some $m \in \mathrm{Max}\, S$.

PROOF. Assume that I is G-maximal. Choose $m \in \mathrm{Max}\, S$, such that $I \subseteq m$. Since I is G-invariant, it follows that $I = {}^gI \subseteq {}^gm$ for all $g \in G$. In particular, $I \subseteq \bigcap_{g\in G} {}^gm = \mathrm{eq}(m)$. Since $\mathrm{eq}(m)$ is a proper ideal of S which is G-invariant, it follows, from the maximality of I, that $I = \mathrm{eq}(m)$.

Conversely, assume that $\mathbf{m} \in \mathrm{Max}\, S$. Let I be a proper G-invariant ideal of S containing $\mathrm{eq}(m)$. We show that $I = \mathrm{eq}(m)$. Let n be a maximal ideal of S containing I. Since I

is G-invariant, it follows that $I \subseteq \mathrm{eq}(n)$. Hence

$$\mathrm{eq}(m) \subseteq I \subseteq \mathrm{eq}(n). \tag{A1.6}$$

It follows, by lemma (A1.10), that m and n are conjugate of one another by an element of G. Hence $\mathrm{eq}(m) = \mathrm{eq}(n)$. From formula (A1.6) it follows that $\mathrm{eq}(m) = I$. $\triangle$

Let I be an ideal of S. Let

$$\mathrm{Stab}_G(I) = \{g \in G \mid {}^gI = I\}. \tag{A1.7}$$

Notice that $\mathrm{Stab}_G(I)$ acts on S/I by R-linear automorphisms.

(A1.18) THEOREM (The Structure of G-Simple Rings).

Assume that S is a G-simple ring. Let $m \in \mathrm{Max}\, S$. Then $\mathrm{eq}(m) = 0$. In particular, the only maximal ideals of S are ${}^gm$ for $g \in G$. Moreover, for $g \in G$, let $S'_g = S/{}^gm$. Then

(1) The rings S'_g are simple, and $S'_{g_1} \cong S'_{g_2}$, for all $g_1, g_2 \in G$.

(2)

$$S \cong \prod_{g \in G/\mathrm{Stab}_G(m)} S'_g.$$

(3) $S^G \cong (S'_e)^{\mathrm{Stab}_G(m)} = (S/m)^{\mathrm{Stab}_G(m)}$.

(4) If S is commutative then the rings S'_g are fields for all $g \in G$. Moreover, if we denote the kernel of the action of $\mathrm{Stab}_G(m)$ on $S'_e = S/m$ by $K(m)$, then S'_e is a Galois extension of $(S'_e)^{\mathrm{Stab}_G(m)} = S^G$ with Galois group $\mathrm{Stab}_G(m)/K(m)$.

PROOF. Let m be a maximal ideal of S. Since the ideal 0 of S is G-maximal it follows that $\mathrm{eq}(m) = 0$. It follows, by lemma (A1.10), that the only maximal ideals of S are the ideals ${}^gm$, for $g \in G$.

(1) Let $g \in G$. It is clear that S'_g is a simple ring. Let $g_1, g_2 \in G$. The R-linear automorphism $c_{g_2 g_1^{-1}}$ induces an R-linear epimorphism

$$S \xrightarrow{c_{g_2 g_1^{-1}}} S \xrightarrow{\pi} S/{}^{g_2}m = S'_{g_2}$$

whose kernel is exactly ${}^{g_1}m$. We conclude that $c_{g_2 g_1^{-1}}$ induces an isomorphism from S'_{g_1} to S'_{g_2}.

(2) Notice that

$$\mathrm{eq}(m) = \bigcap_{g \in G} {}^gm = \bigcap_{g \in G/\mathrm{Stab}_G(m)} {}^gm.$$

Moreover, if

$$g_1\mathrm{Stab}_G(m) \neq g_2\mathrm{Stab}_G(m),$$

then ${}^{g_1}m \neq {}^{g_2}m$. Since ${}^{g}m$ are maximal ideals it follows immediately that

$${}^{g_1}m + {}^{g_2}m = S, \qquad \text{whenever} \qquad g_1\mathrm{Stab}_G(m) \neq g_2\mathrm{Stab}_G(m).$$

It follows easily that

$$S = \frac{S}{\mathrm{eq}(m)} = \frac{S}{\bigcap_{g \in G/\mathrm{Stab}_G(m)} {}^{g}m} \cong \prod_{g \in G/\mathrm{Stab}_G(m)} \frac{S}{{}^{g}m} = \prod_{g \in G/\mathrm{Stab}_G(m)} S'_g.$$

(3) Follows immediately from *(2)*.

(4) See proposition (7.2) [Le1]. $\triangle$

We now investigate the G-equivariant prime ideals of S.

(A1.19) PROPOSITION.

(1) Let P be a G-invariant ideal of S. Then P is G-prime if and only if, whenever I and J are ideals of S such that $I \cdot {}^{g}J \subseteq P$ for all $g \in G$, then $I \subseteq P$ or $J \subseteq P$.

(2) Assume that S is commutative, and let P be a G-invariant ideal of S. Then P is G-prime if and only if, whenever $a,\ b \in S$ are such that $a \cdot c_g(b) \in P$ for all $g \in G$, then $a \in P$ or $b \in P$.

PROOF. *(1)* Assume first that P is G-prime, and let I and J be two ideals of S such that $I \cdot {}^{g}J \subseteq P$ for all $g \in G$. Since P is G-invariant, the above condition is obviously equivalent to ${}^{g_1}I \cdot {}^{g_2}J \subseteq P$ for all $g_1,\ g_2 \in G$. Then

$$\langle I \rangle_G \cdot \langle J \rangle_G = \Big(\sum_{g\in G} {}^{g}I\Big) \cdot \Big(\sum_{g \in G} {}^{g}J\Big) = \sum_{g_1, g_2 \in G} {}^{g_1}I \cdot {}^{g_2}J \subseteq P.$$

Since P is G-prime, it follows that $\langle I \rangle_G \subseteq P$ or $\langle J \rangle_G \subseteq P$. Since $I \subseteq \langle I \rangle_G$ and $J \subseteq \langle J \rangle_G$, it follows that $I \subseteq P$ or $J \subseteq P$.

Conversely, assume that P is a G-invariant ideal such that, whenever I and J are ideals of S with $I \cdot {}^{g}J \subseteq P$ for all $g \in G$, then $I \subseteq P$ or $J \subseteq P$. Let I and J be two G-invariant ideals of S such that $I \cdot J \subseteq P$. Since J is G-invariant, it follows that ${}^{g}J = J$ for all $g \in G$. In particular, $I \cdot {}^{g}J = I \cdot J \subseteq P$ for all $g \in G$. But then $I \subseteq P$ or $J \subseteq P$. This shows that P is G-prime.

(2) Assume that S is commutative. Let P be a G-prime ideal of S. Let a and b be two elements of S such that $a \cdot c_g(b) \in P$ for all $g \in G$. This implies that

$$(Sa) \cdot {}^g(Sb) = (Sa) \cdot S(c_g(b)) = S(a \cdot c_g(b)) \subseteq P \quad \text{for all } g \in G.$$

From *(1)*, it follows that $Sa \subseteq P$ or $Sb \subseteq P$. Hence $a \in P$ or $b \in P$.

Conversely, assume that P is a G-invariant ideal which has the property that, whenever $a,\ b \in S$ are such that $a \cdot c_g(b) \in P$ for all $g \in G$, then $a \in P$ or $b \in P$. Let I and J be two ideals of S such that $I \cdot {}^gJ \subseteq P$ for all $g \in G$. Assume that $I \not\subseteq P$. We show that $J \subseteq P$. Indeed, let $a \in I - P$ and let $b \in J$. Notice that $a \cdot c_g(b) \in I \cdot {}^gJ \subseteq P$ for all $g \in G$. Since $a \notin P$, it follows that $b \in P$. Since b was arbitrary in J, it follows that $J \subseteq P$. $\triangle$

(A1.20) PROPOSITION ([Le1]).

Let S be a G-domain. Then

(1) The non-zero invariant elements of S are not zero divisors. In particular, S^G is an integral domain.

(2) S contains no non-zero nilpotent elements.

(3) S can be written as a finite product $\prod_{i=1}^{n} S_i$ of rings S_i such that the S_i are all isomorphic, and no S_i contains a non-trivial idempotent.

(4) If G_1 is the subgroup of G taking the ring S_1 above to itself, then S_1 is a G_1-domain, and the ring $S_1^{G_1}$ of G_1-invariant elements of S_1 is isomorphic to S^G.

(5) Every element in S_1 satisfies a monic polynomial with coefficients in $S_1^{G_1}$ (and the same applies to S and S^G).

(6) If $S_1^{G_1} = S^G$ is a field and K is the kernel of the action of G_1 on S_1, then S_1 is a Galois extension of $S_1^{G_1}$ with Galois group G_1/K.

PROOF. See [Le1], proposition (7.2). $\triangle$

(A1.21) PROPOSITION.

Let S be a G-domain. Then S is a G-field if and only if S^G is a field.

PROOF. Assume that S is a G-field. It follows, by theorem (A1.18), that S^G is a field.

Conversely, assume that S^G is a field. Let I be a proper G-invariant ideal of S. Since I is proper and S^G is a field, it follows that $I^G = 0$. We show that $I = 0$. Assume that this is not the case and let $x \in I$. Let $F \in S^G[X]$ be the polynomial

$$F(X) = \prod_{g \in G} (X - c_g(x)) = X^{|G|} + \sum_{i=1}^{|G|} a_i \cdot X^{|G|-i}.$$

Since $a_i \in I^G = 0$, it follows that $F(X) = X^{|G|}$. However, x is a root of $F(X)$. Since S is a G-domain, from proposition (A1.20) *(2)* it follows that $x = 0$. $\triangle$

(A1.22) THEOREM (The Structure of G-Prime Ideals of S).

(1) Assume that P is a prime ideal of S. Then the ideal $\text{eq}(P)$ *is G-prime.*

(2) Assume that S is commutative or S is left-noetherian. Then every G-prime ideal of S is of the form $\text{eq}(P)$ *for some prime ideal P of S.*

(3) Let P and Q be two prime ideals of the ring S. Then $\text{eq}(P) = \text{eq}(Q)$ *if and only if $P = {}^gQ$ for some $g \in G$.*

PROOF. *(1)* Let P be a prime ideal of S, and let I and J be two G-invariant ideals of S such that $I \cdot J \subseteq \text{eq}(P)$. Since $\text{eq}(P) \subseteq P$, it follows that $I \cdot J \subseteq P$. Since P is prime, we conclude that $I \subseteq P$ or $J \subseteq P$. Assume, for example, that $I \subseteq P$. Since I is G-invariant, it follows that $I = {}^gI \subseteq {}^gP$ for all $g \in G$. In particular, $I \subseteq \bigcap_{g \in G} {}^gP = \text{eq}(P)$. Hence $\text{eq}(P)$ is G-prime.

(2) CASE 1. Assume that S is commutative. Let I be a G-prime ideal of S. We show that the ideal 0 of the G-domain S/I is $\text{eq}(P)$ for some prime ideal P of S/I. Hence, we assume that S is a G-domain . Let $U = S^G - \{0\}$. It is clear that U is a multiplicative subset of S. Moreover, from proposition (A1.20) *(1)*, we know that U does not contain any zero divisors. In particular, the canonical map

$$j : S \longrightarrow U^{-1}S$$

is injective. Notice that since $U \subseteq S^G$, it follows that G acts on $U^{-1}S$ by R-linear ring automorphism and j is a G-map. Notice also that $U^{-1}S$ is a G-domain with the property that every non-zero element in $(U^{-1}S)^G$ is invertible. From proposition (A1.21), it follows that $U^{-1}S$ is a G-field. Since the ideal 0 of $U^{-1}S$ is G-maximal, it follows, by theorem (A1.17), that $0 = \text{eq}(m)$ for some maximal ideal m of $U^{-1}S$. It is known that every maximal ideal of $U^{-1}S$ is of the form $P \cdot U^{-1}S$ for some prime ideal P of S. Let P be a prime ideal of S such that $m = P \cdot U^{-1}S$. Then, since $U \subset S^G$, it follows that ${}^gm = {}^gP \cdot U^{-1}S$. In particular,

$$0 = \text{eq}(m) = \bigcap_{g \in G} {}^gP \cdot U^{-1}S = \Big(\bigcap_{g \in G} {}^gP\Big) \cdot U^{-1}S = \text{eq}(P) \cdot U^{-1}S,$$

where the third equality above follows from the injectivity of j. Finally, since j is injective and $0 = \text{eq}(P) \cdot U^{-1}S$, it follows that $\text{eq}(P) = 0$.

CASE 2. Assume that S is left-noetherian. Let I be a G-prime ideal of S. Replace S with the G-prime ring S/I. We show that $0 = \mathrm{eq}(P)$ for some P prime ideal of S. Since S is left-noetherian, it follows that $\mathrm{Ass}(S) \neq 0$. Let $P \in \mathrm{Ass}(S)$, and choose a non-zero left ideal N of S such that $\mathrm{Ann}(N') = P$ for all $0 \neq N' \subseteq N$. In particular, $P \cdot N = 0$. If we multiply this relation with an arbitrary element $x \in S$, we conclude that

$$P \cdot (Nx) = 0, \qquad \text{for all } x \in S. \tag{A1.8}$$

Let

$$J = N \cdot S$$

Notice that J is an ideal of S. Moreover, since every element $y \in J$ is of the form $y = \sum_{i=1}^{s} n_i x_i$, for some $n_i \in N$ and $x_i \in S$, it follows easily, from relation (A1.8), that $P \cdot J = 0$. Since $\mathrm{eq}(P) \subseteq P$, it follows that $\mathrm{eq}(P) \cdot J = 0$. Now $\mathrm{eq}(P)$ is invariant, hence

$$0 = {}^{g}0 = {}^{g}\big(\mathrm{eq}(P) \cdot J\big) = {}^{g}\big(\mathrm{eq}(P)\big) \cdot {}^{g}J = \mathrm{eq}(P) \cdot {}^{g}J, \qquad \text{for all } g \in G.$$

Since 0 is a G-prime ideal and $0 \neq N \subseteq J$, it follows, from proposition (A1.19) *(1)*, that $\mathrm{eq}(P) = 0$.

(3) It is clear that if $P = {}^{g}Q$ for some $g \in G$, then $\mathrm{eq}(P) = \mathrm{eq}(Q)$. Conversely assume that $\mathrm{eq}(P) = \mathrm{eq}(Q)$. Since

$$\prod_{g \in G} {}^{g}P \subseteq \bigcap_{g \in G} {}^{g}P = \mathrm{eq}(P) = \mathrm{eq}(Q) \subseteq Q,$$

and Q is prime, it follows that ${}^{g_1}P \subseteq Q$ for some $g_1 \in G$. If we interchange P and Q, we obtain that ${}^{g_2}Q \subseteq P$ for some $g_2 \in G$. Hence

$${}^{g_1}P \subseteq Q \subseteq {}^{g_2^{-1}}P.$$

It is enough to show that ${}^{g_1}P = {}^{g_2^{-1}}P$, or ${}^{g_2 g_1}P = P$. Let $g = g_2 g_1$, and assume that the order of g in G is k. Since ${}^{g}P \subseteq P$, it follows that

$$P = {}^{g^k}P \subseteq {}^{g^{k-1}}P \subseteq \ldots \subseteq {}^{g}P \subseteq P.$$

From the above chain of containments, it follows that ${}^{g^s}P = {}^{g^{s-1}}P$ for all $s \in \mathbb{Z}$. Hence ${}^{g}P = P$. $\triangle$

For our next couple of results we need the following fact from Galois theory.

(A1.23) LEMMA.

Assume that F is a field and G is a finite group acting on F by field automorphisms (not necessarily effectively). Let K be the kernel of the action of G on F. Then

(1) F is a Galois extension of the field F^G with Galois group G/K.

(2) Let $Tr^G = \{\sum_{g \in G} g \cdot a \mid a \in F\}$. Then

$$Tr^G = |K| \cdot F^G.$$

In particular, the trace map

$$tr : F \longrightarrow F^G, \qquad tr(a) = \sum_{g \in G} g \cdot a,$$

is onto if and only if char F *does not divide the order of K. Otherwise, the trace map is identically zero.*

PROOF. See [K], exercise 1, p. 44. $\triangle$

Let S be a commutative ring and let I be an ideal of S. Let $K(I)$ be the kernel of the action of $\mathrm{Stab}_G(I)$ on the ring S/I.

(A1.24) PROPOSITION.

Let S be a G-field and let m be a maximal ideal of S. The trace map

$$tr : S \longrightarrow S^G \qquad a \longmapsto tr(a) = \sum_{g \in G} c_g(a)$$

is onto if and only if the characteristic of the field S/m does not divide the order of $K(m)$. If this condition is not fulfilled then the map tr is identically zero.

PROOF. Since S^G is a field it follows that tr is either onto or identically zero. It is clear that tr is surjective if and only if the $R[G]$-algebra S is projective. From proposition (9.1.10), it follows that tr is onto if and only if the $R[\mathrm{Stab}_G(m)]$-algebra S/m is projective. This last condition is equivalent to the fact that the trace map

$$tr : S/m \longrightarrow (S/m)^{\mathrm{Stab}_G(m)} \qquad tr(a) = \sum_{g \in \mathrm{Stab}_G(m)} c_g(a)$$

is onto. Now the assertion of the proposition follows from theorem (A1.18) *(4)* and lemma (A1.23). $\triangle$

(A1.25) PROPOSITION.

Let S be a G-domain. Suppose that $0 = \mathrm{eq}(p)$ for some prime ideal p of S (see theorem (A1.22) (2)). Then the trace map

$$tr : S \longrightarrow S^G \qquad tr(a) = \sum_{g \in G} c_g(a)$$

is identically zero if and only if the characteristic of the domain S/p divides the order of $K(p)$.

PROOF. Let $U = S^G - \{0\}$. By proposition (A1.20) *(2)* we know that U does not contain zero divisors. It is easy to see that $U^{-1}S$ is a G-field. Moreover, the ideal $U^{-1}p = p \cdot U^{-1}(S)$ is maximal in $U^{-1}(p)$. Notice that the subgroup $\mathrm{Stab}_G(U^{-1}p)$ is $\mathrm{Stab}_G(p)$ and that the kernel of the action of $\mathrm{Stab}_G(p)$ on $U^{-1}(S/p)$ coincides with $K(p)$. Moreover, the trace of G on S is identically zero if and only if the trace of G on $U^{-1}S$ is identically zero. Now the result follows from proposition (A1.24). Δ

We now investigate the G-tertiary and the G-primary submodules of a G-equivariant left-S-module.

(A1.26) DEFINITION. Assume that S is a left-noetherian ring and let M be a G-equivariant left-S-module. An ideal I of S is called *G-associated to M* if there exists a non-zero G-invariant submodule N of M such that $I = \mathrm{Ann}(N')$ for all non-zero G-invariant submodules N' of N.

Notice that if I is G-associated to M, then I must be G-invariant.

(A1.27) PROPOSITION.

If I is G-associated to M, then I is G-prime.

PROOF. Assume that J_1, J_2 are two G-invariant ideals of S such that $J_1 \cdot J_2 \subseteq P$, but $J_2 \not\subseteq P$. Choose N a non-zero G-invariant submodule of M such $\mathrm{Ann}(N') = I$, for all $0 \neq N' \subseteq N$ with N' G-invariant. Since $J_2 \not\subseteq I$, and $I = \mathrm{Ann}(N)$, it follows that $J_2 \cdot N \neq 0$. Since $J_2 \cdot N$ is a G-invariant submodule of N and $J_1 \cdot J_2 \subseteq I = \mathrm{Ann}(N)$, it follows that $J_1 \subseteq \mathrm{Ann}(J_2 \cdot N) = I$. This shows that I is G-prime. Δ

Let $\mathrm{Ass}_G(M)$ be the set of all G-prime ideals associated to M. Notice that, since S was assumed left-noetherian, the set of annihilators of non-zero G-invariant submodules of M has maximal elements. In particular, if M is non-zero, then $\mathrm{Ass}_G(M) \neq \emptyset$. We investigate the relationship between $\mathrm{Ass}_G(M)$ and $\mathrm{Ass}(M)$ (See [St], p. 160-162 for the properties of Ass).

Recall that an S-module N is called *cotertiary* if $\mathrm{Ass}(N)$ has only one member. If $\mathrm{Ass}(N) = \{P\}$ we refer to N as being *P-cotertiary*. In this case, we denote P by $\mathrm{ass}(N)$. Now if M is an S-module and $P \in \mathrm{Ass}(M)$ then it follows easily that M has a P-cotertiary submodule N. We begin with the following lemma.

(A1.28) LEMMA.

Let S be a left-noetherian ring and let M be a left-S-module. Assume that $P_1, \ldots, P_n$ are distinct ideals in $\mathrm{Ass}(M)$. Let $N_1, \ldots, N_n$ be cotertiary submodules of M such that $\mathrm{ass}(N_i) = P_i$. Let

$$N = \sum_{i=1}^{n} N_i.$$

Then the above sum is direct and $\mathrm{Ass}(N) = \{P_i \mid i = 1, \ldots, n\}$.

PROOF. Let $n = 2$. If $N_1 \cap N_2 \neq 0$, then

$$\mathrm{Ass}(N_1 \cap N_2) \subseteq \mathrm{Ass}(N_1) \cap \mathrm{Ass}(N_2) = \{P_1\} \cap \{P_2\} = \emptyset.$$

Hence $N_1 \cap N_2 = 0$. It follows that

$$\mathrm{Ass}(N) = \mathrm{Ass}(N_1 \oplus N_2) = \mathrm{Ass}(N_1) \cup \mathrm{Ass}(N_2) = \{P_1, P_2\}.$$

The general case follows easily by induction on n. $\triangle$

(A1.29) THEOREM.

Let S be a left-noetherian ring and let M be a G-equivariant left-S-module. Then

$$\mathrm{Ass}_G(M) = \{\mathrm{eq}(P) \mid P \in \mathrm{Ass}(M)\}.$$

PROOF. Let $I \in \mathrm{Ass}_G(M)$. Choose a non-zero G-invariant submodule N of M such that $I = \mathrm{Ann}(N')$ whenever N' is a non-zero G-invariant submodule of N. Since $N \neq 0$, it follows that $\mathrm{Ass}(N) \neq \emptyset$. Choose $P \in \mathrm{Ass}(N)$, and let N_1 be a submodule of N such that $P = \mathrm{Ann}(N_1)$. It is clear that ${}^gP = \mathrm{Ann}({}^gN_1)$ for all $g \in G$. Hence

$$\mathrm{Ann}\big(\langle N_1\rangle_G\big) = \mathrm{Ann}\Big(\sum_{g\in G} {}^gN_1\Big) = \bigcap_{g\in G} \mathrm{Ann}({}^gN_1) = \bigcap_{g\in G} {}^gP = \mathrm{eq}(P).$$

On the other hand, since $N_1 \subseteq N$ and N is G-invariant, it follows that ${}^gN_1 \subseteq {}^gN = N$ for all $g \in G$. In particular, $\langle N_1\rangle_G$ is a non-zero G-invariant submodule of N. We conclude that $I = \mathrm{Ann}\big(\langle N_1\rangle_G\big) = \mathrm{eq}(P)$. Hence

$$\mathrm{Ass}_G(M) \subseteq \{\mathrm{eq}(P) \mid P \in \mathrm{Ass}(M)\}. \qquad (A1.9)$$

Conversely, let $P \in \mathrm{Ass}(M)$. We construct a non-zero G-invariant submodule N of M such that

$$\mathrm{Ass}(N) = \{{}^gP \mid g \in G\}. \tag{A1.10}$$

We distinguish 2 cases.

CASE I. *Assume that* $\mathrm{Stab}_G(P) = G$.

Let L_1 be a P-cotertiary submodule of M. Since P is G-invariant, it follows that $\mathrm{ass}({}^gL_1) = P$ for all $g \in G$. Let $n = |G|$ and suppose that $G = \{g_1, \ldots, g_n\}$. Assume that $e = g_1$. Let

$$\mathcal{X}_2(L_1) = \{(i,\ j) \mid {}^{g_i}L_1 \cap {}^{g_j}L_1 = 0\}$$

Assume first that $\mathcal{X}_2(L_1) \neq \emptyset$. Up to conjugation we may assume that $(1, k) \in \mathcal{X}_2(L_1)$. Since $L_1 \cap {}^{g_k}L_1 = 0$, we conclude that if

$$L_1' = L_1 + {}^{g_k}L_1,$$

then the above sum is direct; hence $\mathrm{Ass}(L_1') = \mathrm{Ass}(L_1) \cup \mathrm{Ass}({}^{g_k}L_1) = \{P\}$. Notice that if $(i,\ j) \notin \mathcal{X}_2(L_1)$ then $0 \neq {}^{g_i}L_1 \cap {}^{g_j}L_1 \subseteq {}^{g_i}L_1' \cap {}^{g_j}L_1'$. Moreover, $0 \neq {}^{g_k}L_1 \subseteq {}^{e}L_1' \cap {}^{g_k}L_1'$. It follows that $|\mathcal{X}_2(L_1')| < |\mathcal{X}_2(L_1)|$. Repeating this argument finitely many times we end up with a P-cotertiary submodule L_2 such that ${}^gL_2 \cap {}^hL_2 \neq 0$ for all $g,\ h \in G$. Of course, if $\mathcal{X}_2(L_1)$ was empty then we can set $L_2 = L_1$.

We do one more step. Let

$$\mathcal{X}_3(L_2) = \{(i,\ j,\ k) \mid {}^{g_i}L_2 \cap {}^{g_j}L_2 \cap {}^{g_k}L_3 = 0\}.$$

Assume first that $\mathcal{X}_3(L_2) \neq 0$. Up to conjugation, we may assume that $(1,\ j,\ k) \in \mathcal{X}_3(L_2)$. Let

$$L_2' = L_2 + {}^{g_j}L_2 \cap {}^{g_k}L_2.$$

The above sum is direct, therefore

$$\mathrm{Ass}(L_2') = \mathrm{Ass}(L_2) + \mathrm{Ass}({}^{g_j}L_2 \cap {}^{g_k}L_2) = \{P\}$$

(because ${}^{g_j}L_2 \cap {}^{g_k}L_2$ is a non-zero submodule of ${}^{g_j}L_2$; hence it is P-cotertiary). Notice that if $(i_1,\ j_1,\ k_1) \notin \mathcal{X}_3(L_2)$ then

$$0 \neq {}^{g_{i_1}}L_2 \cap {}^{g_{j_1}}L_2 \cap {}^{g_{k_1}}L_2 \subseteq {}^{g_{i_1}}L_2' \cap {}^{g_{j_1}}L_2' \cap {}^{g_{k_1}}L_2'$$

and

$$0 \neq {}^{g_i}L_2 \cap {}^{g_j}L_2 \subseteq L_2' \cap {}^{g_i}L_2' \cap {}^{g_j}L_2'.$$

Hence $|\mathcal{X}_3(L_2')| < |\mathcal{X}_3(L_2)|$. After finitely many steps we end up with a P-cotertiary submodule L_3 such that $\mathcal{X}_3(L_3) = \emptyset$.

Now it is clear that by iterating this argument we end up with a P-cotertiary submodule L_n of M such that

$$\mathcal{X}_n(L_n) = \{(i_1, \ ..., \ i_n) \mid \bigcap_{j=1}^{n} {}^{g_{i_j}}L_n = 0\} = \emptyset.$$

This submodule has the property that $\mathrm{eq}(L_n) = \bigcap_{g \in G} {}^{g}L_n \neq \emptyset$. Let $N = \mathrm{eq}(L_n)$. Then N is a non-zero G-invariant submodule of M which satisfies equation (A1.10).

CASE II. *Assume that* $H = \mathrm{Stab}_G(P) < G$.

It follows, by case I, that M has a submodule L which is H-invariant and P-cotertiary. Hence ${}^{g}L$ is ${}^{g}P$-cotertiary. It follows, by lemma (A1.28), that if we let

$$N = \sum_{g \in G/H} {}^{g}L,$$

then $\mathrm{Ass}(N) = \{{}^{g}P \mid g \in G\}$. It is clear that N is G-invariant. Hence N satisfies equation (A1.10).

Let now N be a G-invariant submodule of M satisfying equation (A1.10). We show that $\mathrm{Ass}_G(N) = \{\mathrm{eq}(P)\}$. Indeed, let $\mathrm{eq}(Q)$ (see theorem (A1.22)) be a G-associated ideal of N. Let N_1 be a G-invariant submodule of N such that $\mathrm{Ann}(N_1') = \mathrm{eq}(Q)$ for all non-zero G-invariant submodules N_1' of N. It is clear that $\mathrm{Ass}(N_1) \subseteq \mathrm{Ass}(N) = \{{}^{g}P \mid g \in G\}$, and since N_1 is non-zero and G-invariant, it follows that

$$\mathrm{Ass}(N_1) = \{{}^{g}P \mid g \in G\}.$$

Since $\mathrm{eq}(Q) \in \mathrm{Ass}_G(N_1)$ it follows, by containment $(A1.9)$, that $\mathrm{eq}(Q) = \mathrm{eq}(P)$. Since P was arbitrary in $\mathrm{Ass}(M)$, it follows that

$$\{\mathrm{eq}(P) \mid P \in \mathrm{Ass}(M)\} \subseteq \mathrm{Ass}_G(M). \qquad \triangle$$

We record the following properties of Ass_G.

(A1.30) PROPOSITION.

Let S be a left-noetherian ring. (1) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence in G-S-Mod then

$$\mathrm{Ass}_G(L) \subseteq \mathrm{Ass}_G(M) \subseteq \mathrm{Ass}_G(L) \cup \mathrm{Ass}_G(N).$$

(2) If M_i are G-equivariant left-S-modules then

$$\mathrm{Ass}\Big(\bigoplus_{i\in I} M_i\Big) = \bigcup_{i\in I} \mathrm{Ass}_G(M_i).$$

(3) If I is a G-prime ideal of S, then $\mathrm{Ass}_G(S/I) = \{I\}$.

PROOF. The above relations follow immediately from the classical properties of Ass via theorems (A1.29) and (A1.22). $\triangle$

(A1.31) DEFINITION. A G-equivariant left-S-module M is called *G-cotertiary*, if $\mathrm{Ass}_G(M)$ has only one member. When $\mathrm{Ass}_G(M) = \{\mathrm{eq}(P)\}$ we denote $\mathrm{eq}(P)$ by $\mathrm{ass}(M)$. We sometimes refer to M as being $\mathrm{eq}(P)$-G-cotertiary. If M is a G-equivariant left-S-module and N is a G-invariant submodule of M, then N is called *G-tertiary in M* if M/N is G-cotertiary. If $\mathrm{ass}(M/N) = \mathrm{eq}(P)$ we refer to N as being *$\mathrm{eq}(P)$-G-tertiary in M*.

(A1.32) PROPOSITION.

Let S be a left-noetherian ring and let M be a G-equivariant left-S-module. Let I be a G-prime ideal of S. Let $(N_i)_{i\in\Gamma}$ be a family of I-G-tertiary submodules in M. Then $\bigcap_{i\in\Gamma} N_i$ is I-G-tertiary in M as well.

(A1.33) THEOREM (Characterization Theorem for G-Cotertiary Modules).

Let S be a left-noetherian ring and let M be a G-equivariant left-S-module. Let P be prime ideal of S and let $I = \mathrm{eq}(P)$. Then M is I-G-cotertiary if and only if

$$\mathrm{Ass}(M) = \{{}^gP \mid g \in G\}.$$

PROOF. Immediate consequence of theorem (A1.29). $\triangle$

(A1.34) THEOREM (Characterization Theorem for G-Tertiary Submodules in M).

Let S be a left-noetherian ring and let M be a G-equivariant left-S-module. Let P be a prime ideal of S. Denote $\mathrm{eq}(P)$ by I. Then

(1) Let N be a G-invariant submodule of M. Then N is I-G-tertiary in M if and only if

$$\mathrm{Ass}(M/N) = \{{}^gP \mid g \in G\}.$$

(2) Let N' be a P-tertiary submodule in M. Then $\text{eq}(N')$ *is an I-G-tertiary submodule in M.*

(3) Assume that M is finitely generated, and suppose that N is an I-G-tertiary submodule in M. Then there exists a P-tertiary submodule N' in M such that $N = \text{eq}(N')$. Moreover, if we let

$$\text{Stab}_G(P) = \{g \in G \mid {}^gP = P\},$$

then N' can be chosen such that N' is $\text{Stab}_G(P)$*-invariant.*

PROOF. *(1)* Follows immediately from theorem (A1.29).

(2) We know that $\text{Ass}(M/N') = \{P\}$. It follows immediately that $\text{Ass}(M/{}^gN') = \{{}^gP\}$ for all $g \in G$. From the canonical injection

$$\frac{M}{\text{eq}(N')} = \frac{M}{\bigcap_{g\in G} {}^gN'} \longrightarrow \bigoplus_{g\in G} \frac{M}{{}^gN'},$$

it follows that

$$\text{Ass}(M/\text{eq}(N')) \subseteq \text{Ass}\Big(\bigoplus_{g\in G} \frac{M}{{}^gN'}\Big) = \bigcup_{g\in G} \text{Ass}(M/{}^gN') = \{{}^gP \mid g \in G\}. \qquad (A1.11)$$

Since $M/\text{eq}(N')$ is G-equivariant, it follows that $\text{Ass}(M/\text{eq}(N'))$ is invariant under the action of G. In particular, containment (A1.11) is, in fact, an equality. From *(1)*, we conclude that $\text{eq}(N')$ is an $\text{eq}(P)$-G-tertiary submodule in M.

(3) Assume that N is an $\text{eq}(P)$-G-tertiary submodule in M. From *(1)* it follows that

$$\text{Ass}(M/N) = \{{}^gP \mid g \in G\}.$$

Let $H = \text{Stab}_G(P)$. Since M is finitely generated, it follows, from the classical tertiary decomposition theorem, that

$$N = \bigcap_{gH\in G/H} N'_{gH}$$

where N_{gH} is a ${}^gP$-tertiary submodule in M, for all $gH \in G/H$. Let

$$N' = \bigcap_{g\in G} {}^{g^{-1}}N_{gH}. \qquad (A1.12)$$

Since N is G-invariant and $N \subseteq N_{gH}$ for all $g \in G$, it follows that $N \subseteq N'$. Moreover, since N_{gH} is ${}^gP$-tertiary in M, it follows that ${}^{g^{-1}}N_{gH}$ is P-tertiary in M for all $g \in G$.

Since intersections of P-tertiary submodules in M are P-tertiary, it follows that N' is P-tertiary in M. From formula (A1.12), it follows immediately that N' is H-invariant. We show that $N = \mathrm{eq}(N')$. We know that $N \subseteq N'$. Since N is G-invariant, it follows that $N \subseteq \mathrm{eq}(N')$. On the other hand, for $g \in G$, ${}^{g}N' \subseteq N_{gH}$ (see formula (A1.12)). Hence $\mathrm{eq}(N') \subseteq \bigcap_{gH \in G/H} N_{gH} = N$. Therefore $N = \mathrm{eq}(N')$. $\triangle$

(A1.35) THEOREM (The G-Tertiary Decomposition Theorem).

Let S be a left-noetherian ring and let M be a finitely generated G-equivariant left-S-module. Let N be a G-invariant submodule of M. Assume that

$$\mathrm{Ass}_G(M/N) = \{\mathrm{eq}(P_i) \mid i = 1, \ \ldots, \ n\}.$$

Then N can be written as an intersection

$$N = L_1 \cap \ldots \cap L_n$$

such that:

(1) The submodules L_i are $\mathrm{eq}(P_i)$-G-tertiary in M.

(2) The decomposition is irredundant.

(3) The submodule L_i can be written as $L_i = \mathrm{eq}(L_i')$ for some P_i-tertiary submodule L_i' in M which is $\mathrm{Stab}_G(P_i)$-invariant.

If $N = N_1 \cap \ldots \cap N_m$ is another tertiary decomposition with the properties (1)-(3) above then $m = n$ and

$$\{\mathrm{ass}(M/L_i)\} = \{\mathrm{ass}(M/N_i)\}.$$

PROOF. Obvious consequence of the classical tertiary decomposition theorem and theorem (A1.34). $\triangle$

(A1.36) DEFINITION. Let I be a G-prime ideal of S. A G-equivariant left-S-module M is called *I-G-coprimary* if M is I-G-cotertiary and $I^n \subseteq \mathrm{Ann}(M)$ for some $n \geq 1$. If M is a G-equivariant left-S-module and N is a G-invariant submodule of M, then N is *I-G-primary in M* if M/N is I-G-coprimary.

(A1.37) THEOREM.

Let S be commutative noetherian ring. Let P be a prime ideal of S and let $I = \mathrm{eq}(P)$. Let M be a finitely generated G-equivariant S-module. Then:

(1) The module M is I-G-cotertiary if and only if it is I-G-coprimary. In particular, M is I-G-coprimary if and only if

$$\mathrm{Ass}(M) = \{{}^gP \mid g \in G\}.$$

(2) A G-invariant submodule N of M is I-G-primary in M if and only if

$$\mathrm{Ass}(M/N) = \{{}^gP \mid g \in G\}.$$

(3) (The structure of G-primary submodules in M).

Let N' be a P-primary submodule in M. Then $\mathrm{eq}(N')$ is an I-G-primary submodule in M. Conversely, every I-G-primary submodule N in M is of the form $\mathrm{eq}(N')$ for some P-primary submodule N' of M. The submodule N' above can be choosen to be $\mathrm{Stab}_G(P)$-invariant.

(4) (The G-primary decomposition property).

Let N be a G-invariant submodule of M. Let $\mathrm{Ass}_G(M/N) = \{\mathrm{eq}(P_i) \mid i = 1, 2, ..., n\}$. Then N can be written as an intersection

$$N = L_1 \cap ... \cap L_n$$

such that:

(4.1) L_i are $\mathrm{eq}(P_i)$-G-primary submodules in M.

(4.2) The decomposition is irredundant.

(4.3) L_i can be written as $L_i = \mathrm{eq}(L_i')$ for some P_i-primary submodule L_i' in M which is $\mathrm{Stab}_G(P_i)$ invariant.

If $N = N_1 \cap ... \cap N_m$ is another primary decomposition with the properties (4.1)-(4.3) above then $m = n$ and

$$\{\mathrm{ass}(M/L_i)\} = \{\mathrm{ass}(M/N_i)\}.$$

PROOF. *(1)* Let M be I-G-cotertiary. In particular, the submodule 0 of M is I-G-tertiary in M. According to theorem (A1.34), $0 = \mathrm{eq}(N)$, for some submodule N of M which is P-tertiary in M. Since S is commutative and M is finitely generated, it follows that

N is, in fact, P-primary in M. In particular, there exists $n \geq 1$ such that $P^n \subseteq \mathrm{Ann}(M/N)$. We conclude that

$$({}^gP)^n = {}^g(P^n) \subseteq {}^g\Big(\mathrm{Ann}(M/N)\Big) = \mathrm{Ann}(M/{}^gN), \qquad \text{for all } g \in G.$$

It follows that

$$I^n = \big(\mathrm{eq}(P)\big)^n \subseteq \bigcap_{g\in G} ({}^gP)^n \subseteq \bigcap_{g\in G} \mathrm{Ann}(M/{}^gN) = \mathrm{Ann}(M/\mathrm{eq}(N)) = \mathrm{Ann}(M).$$

Hence M is I-G-coprimary.

(2)-(4) follow from *(1)* and theorems (A1.34) and (A1.35). △

(A1.38) PROPOSITION.

Let S be a commutative noetherian ring. A finitely generated G-equivariant S-module M is G-coprimary if and only if M satisfies the following property:

-if $a \in S$, and $0 \neq m \in M$ are such that $a \cdot c_g(m) = 0$ for all $g \in G$, then $a^n \in \mathrm{Ann}(M)$ for some $n \geq 1$.

If M is finitely generated and G-coprimary then

$$\mathrm{ass}_G(M) = \sqrt{\mathrm{Ann}(M)}.$$

PROOF. Assume that M is I-G-coprimary. Let $0 \neq m \in M$ and $a \in S$ be such that $a \cdot c_g(m) = 0$ for all $g \in G$. Let $N = \langle m \rangle_G = \sum_{g\in G} S\big(c_g(m)\big)$. Then N is an I-G-cotertiary module. Hence $a \in \mathrm{Ann}(N) \subseteq I$. Since M is I-G-coprimary, it follows that $I^n \subseteq \mathrm{Ann}(M)$ for some $n \geq 1$. We conclude that $a^n \in I^n \subseteq \mathrm{Ann}(M)$.

Conversely, assume that M has the property that, whenever $a \in S$ and $0 \neq m \in M$ are such that $a \cdot c_g(m) = 0$ for all $g \in G$, then $a^n \in \mathrm{Ann}(M)$ for some $n \geq 1$. Choose $I \in \mathrm{Ass}_G(M)$. Let N be a non-zero G-invariant submodule of M such that $I = \mathrm{Ann}(N)$. Choose a non-zero element x of N. Then $a \cdot c_g(x) = 0$ for all $g \in G$ and $a \in I$. It follows that, for every $a \in I$, $a^{k_a} \in \mathrm{Ann}(M)$ for some $k_a \geq 1$ (depending on a). Since S is noetherian, we may assume that I is generated by some elements $a_1, \ldots, a_s$ of I. Let $k = s \cdot \max(k_{a_j} \mid 1 \leq j \leq s)$. Then one can easily show that $I^k \subseteq \mathrm{Ann}(M)$. Hence $I \subseteq \sqrt{\mathrm{Ann}(M)}$. Since, on the other hand, $I \in \mathrm{Ass}_G(M)$, it follows that $\mathrm{Ann}(M) \subseteq I$. We conclude that $\sqrt{\mathrm{Ann}(M)} \subseteq I$. From the above argument, it follows that $I = \sqrt{\mathrm{Ann}(M)}$. In particular, $\sqrt{\mathrm{Ann}(M)}$ is a G-prime ideal, and it is the only ideal in $\mathrm{Ass}_G(M)$. It follows that M is I-G-cotertiary. Since M is finitely generated, it follows, from theorem (A1.37) *(1)*, that M is I-G-coprimary. $\triangle$

A2. MISCELLANEOUS RESULTS.

Throughout this section S is a ring with unity 1_S and M is left-S-module.

(A2.1) PROPOSITION.

Assume that $N_1, \ldots, N_k$ *are submodules of* M *such that* M/N_i *is left-noetherian (left-artinian) for* $1 \leq i \leq k$. *Then*

$$\frac{M}{\bigcap_{i=1}^{k} N_i}$$

is also left-noetherian (left-artinian).

PROOF. Since M/N_i are left-noetherian (left-artinian) for $1 \leq i \leq k$, so is their direct sum. Now the proposition follows from the fact that there exists a canonical injection

$$\frac{M}{\bigcap_{i=1}^{k} N_i} \longrightarrow \bigoplus_{i=1}^{k} \frac{M}{N_i}. \qquad \triangle$$

(A2.2) COROLLARY.

Assume that $I_1, \ldots, I_k$ *are ideals of* S *such that the rings* S/I_j *are left-noetherian (left-artinian), for* $1 \leq j \leq k$. *Then the ring*

$$\frac{S}{\bigcap_{j=1}^{k} I_j}$$

is also left-noetherian (left-artinian).

PROOF. Immediate consequence of proposition (A2.1). $\triangle$

From now on all rings that appear are commutative and unitary.

(A2.3) DEFINITION. Assume that S_2 is a commutative unitary ring and that S_1 is a unitary subring of S_2. Then S_2 is called an *integral extension of* S_1 if every element x of S_2 satisfies a monic polynomial equation of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0, \qquad \text{for some } a_i \in S_1,\ 1 \leq i \leq n.$$

(A2.4) EXAMPLE. Let S be a commutative ring and let G be a finite group acting on S by ring automorphisms. Then S is integral over S^G because every $x \in S$ satisfies the monic polynomial equation $F(x) = 0$, where

$$F(X) = \prod_{g \in G} (X - c_g(x)) \in S^G[X].$$

(A2.5) DEFINITION. Let S_2 be a commutative unitary ring and let S_1 be a unitary subring of S_2. Let P be a prime ideal of S_1 and let P' be a prime ideal of S_2. We say that the ideal P' is *lying over* P if $P' \cap S_1 = P$.

(A2.6) THEOREM.

Let S_2 be an integral extension of S_1 and P be a prime ideal of S_1.

(1) There exists a prime ideal P' of S_2 lying over P.

(2) There are no inclusions between the prime ideals of S_2 lying over P.

PROOF. See [M], theorem 9.3, p. 66. $\triangle$

(A2.7) THEOREM (The Going Up Theorem).

Let S_2 be an integral extension of S_1, P be a prime ideal of S_1 and P' be a prime ideal of S_2 lying over P. Assume that Q is a prime ideal of S_1 such that $P \subseteq Q$. Then there exists a prime ideal Q' of S_2 such that $P' \subseteq Q'$ and Q' sits over Q.

PROOF. See [M], theorem 9.4, p. 68 . $\triangle$

(A2.8) THEOREM (Eakin-Nagata Theorem).

Let S_2 be a commutative noetherian ring and S_1 be a subring of S_2 such that S_2 is finitely generated over S_1 as an S_1-module. Then S_1 is noetherian.

PROOF. See [M], theorem 3.7, p. 18. $\triangle$

(A2.9) THEOREM.

Let S_2 be an commutative artinian ring and S_1 be a subring of S_2 such that S_2 is finitely generated over S_1 as an S_1-module. Then S_1 is artinian.

PROOF. Since S_2 is artinian, it follows that S_2 is noetherian as well. Since S_2 is finitely generated over S_1 as an S_1-module, it follows, from theorem (A2.8), that S_1 is noetherian. We show that the only prime ideals of S_1 are maximal. Indeed, assume that $P \subset Q$ are prime ideals of S_1. Since S_2 is finitely generated over S_1 as an S_1-module, it follows that S_2 is, in particular, integral over S_1. From theorems (A2.6) and (A2.7) it follows that there exists prime ideals $P' \subset Q'$ of S_2 such that $P' \cap S_1 = P$ and $Q' \cap S_1 = Q$. However, this contradicts the fact that S_2 is artinian. Hence, the only prime ideals of S_1 are maximal. Moreover, since S_2 is artinian, it follows that Max S_2 is finite. From theorem (A2.6) it follows that Max S_1 is finite as well. Let Max $S_1 = \{m_1, \ldots, m_k\}$. Then $Nil(S_1) = Jac(S_1) = \bigcap_{i=1}^{k} m_i$. Since S_1 is noetherian, it follows that $Nil(S_1)$ is finitely generated. In particular, $Jac(S_1)^n = 0$ for some $n \geq 1$. Now notice that

$$\frac{S_1}{Jac(S_1)} = \prod_{i=1}^{k} \frac{S_1}{m_i}$$

is a product of finitely many fields. Hence it is artinian (in fact semisimple artinian). Since S_1 is noetherian, it follows that

$$\frac{Jac(S_1)}{Jac(S_1)^2}, \frac{Jac(S_1)^2}{Jac(S_1)^3}, \ldots, \frac{Jac(S_1)^{n-1}}{Jac(S_1)^n} = Jac(S_1)^{n-1}$$

are finitely generated $S_1/Jac(S_1)$-modules, hence artinian. Now it follows easily that S_1 is artinian. $\triangle$

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